On H. Weyl and J. Steiner Polynomials

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Abstract. The paper deals with root location problems for two classes of univariate polynomials both of geometric origin. The first class discussed, the class of Steiner polynomial, consists of polynomials, each associated with a compact convex set $V \subset \mathbb{R}^n$. A polynomial of this class describes the volume of the set $V + tB^n$ as a function of t, where t is a positive number and B^n denotes the unit ball in $\mathbb{R}^n$. The second class, the class of Weyl polynomials, consists of polynomials, each associated with a Riemannian manifold $\mathcal{M}$, where $\mathcal{M}$ is isometrically embedded with positive codimension in $\mathbb{R}^n$. A Weyl polynomial describes the volume of a tubular neighborhood of its associated $\mathcal{M}$ as a function of the tube's radius. These polynomials are calculated explicitly in a number of natural examples such as balls, cubes, squeezed cylinders. Furthermore, we examine how the above mentioned polynomials are related to one another and how they depend on the standard embedding of $\mathbb{R}^n$ into $\mathbb{R}^m$ for $m > n$. We find that in some cases the real part of any Steiner polynomial root will be negative. In certain other cases, a Steiner polynomial will have only real negative roots. In all of this cases, it can be shown that all of a Weyl polynomial's roots are simple and, furthermore, that they lie on the imaginary axis. At the same time, in certain cases the above pattern does not hold.

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Erasmus Darwin, the nephew of the great scientist Charles Darwin, believed that sometimes one should perform the most unusual experiments. They usually yield no results but when they do … . So once he played trumpet in front of tulips for the whole day. The experiment yielded no results.

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1. H. Weyl and J. Steiner polynomials.

Let $\mathcal{M}$ be a smooth manifold,

$$\dim \mathcal{M} = n,$$

which is embedded injectively into the Euclidean space of a higher dimension, say $n+p$, $p>0$. We identify $\mathcal{M}$ with the image of this embedding, so

$$\mathcal{M} \subset \mathbb{R}^{n+p}.$$

For $x \in \mathcal{M}$, let $\mathcal{N}_x$ be the normal subspace to $\mathcal{M}$ at the point x. $\mathcal{N}_x$ is an affine subspace of the ambient space $\mathbb{R}^{n+p}$,

$$\dim \mathcal{N}_x = p.$$

For $t > 0$, let

$$D_x(t) = \{y \in \mathcal{N}_x : \operatorname{dist}(y,x) \leq t\}, \tag{1.1}$$

where $\operatorname{dist}(y,x)$ is the Euclidean distance between x and y. If the manifold $\mathcal{M}$ is compact, and $t>0$ is small enough, then

$$D_{x_1}(t) \cap D_{x_2}(t) = \emptyset \quad \text{for} \quad x_1 \in \mathcal{M}, x_2 \in \mathcal{M}, x_1 \neq x_2. \tag{1.2}$$

Definition 1.1. The set

$$\mathfrak{T}_{\mathcal{M}}^{\mathbb{R}^{n+p}}(t) \stackrel{\text{def}}{=} \bigcup_{x\in\mathcal{M}} D_x(t) \tag{1.3}$$

is said to be the tube neighborhood of the manifold $\mathcal{M}$, or the tube around $\mathcal{M}$. The number t is said to be the radius of this tube.

Is it clear that for manifolds $\mathcal{M}$ *without boundary*,

$$\mathfrak{T}_{\mathcal{M}}^{\mathbb{R}^{n+p}}(t) = \{x \in \mathbb{R}^{n+p} : \operatorname{dist}(x, \mathcal{M}) \le t\}, \tag{1.4}$$

where $\operatorname{dist}(x,\mathcal{M})$ is the Euclidean distance from x to $\mathcal{M}$. Thus, for manifolds without boundary, the equality (1.4) could also be taken as a definition of the tube $\mathfrak{T}_{\mathcal{M}}(t)$. However, for manifolds $\mathcal{M}$ *with boundary* the sets $\mathfrak{T}_{\mathcal{M}}^{\mathbb{R}^{n+p}}(t)$ defined by (1.3) and (1.4) do not coincide. In this, more general, case the tube around $\mathcal{M}$ should be defined by (1.3), but not by (1.4). Hermann Weyl, [Wey1], obtained the following result, which is the starting point of our work:

Theorem (H.Weyl). *Let $\mathcal{M}$, $\dim\mathcal{M} = n$, be a smooth compact manifold, with or without boundary, which is embedded in the Euclidean space $\mathbb{R}^{n+p}$, $p \ge 1$.*

I. If $t > 0$ is small enough[1]*, then the $(n+p)$-dimensional volume of the tube $\mathfrak{T}_{\mathcal{M}}^{\mathbb{R}^{n+p}}(t)$ around $\mathcal{M}$, considered as a function of the radius t of this tube, is a polynomial of the form*

$$\operatorname{Vol}_{n+p}(\mathfrak{T}_{\mathcal{M}}^{\mathbb{R}^{n+p}}(t)) = \omega_p\, t^p \Big(\sum_{l=0}^{[\frac{n}{2}]} u_{2l,p}(\mathcal{M}) \cdot t^{2l}\Big), \tag{1.5}$$

where

$$\omega_p = \frac{\pi^{p/2}}{\Gamma(\frac{p}{2}+1)} \tag{1.6}$$

is the p-dimensional volume of the unit p-dimensional ball.

II. The coefficients $u_{2l,p}(\mathcal{M})$ depend on p as follows:

$$u_{2l,p}(\mathcal{M}) = \frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)} \cdot w_{2l}(\mathcal{M}), \quad 0 \le l \le \left[\tfrac{n}{2}\right], \tag{1.7}$$

where the values $w_{2l}(\mathcal{M})$, $0 \le l \le [\frac{n}{2}]$, may be expressed only in terms of the intrinsic metric[2] of the manifold $\mathcal{M}$. *In particular, the constant term $u_{0,p}(\mathcal{M}) = w_0(\mathcal{M})$ is the n-dimensional volume of $\mathcal{M}$:*

$$w_0(\mathcal{M}) = \operatorname{Vol}_n(\mathcal{M}). \tag{1.8}$$

H. Weyl, [Wey1], expressed the coefficients $w_{2l}(\mathcal{M})$ as integrals of certain rather complicated curvature functions of the manifold $\mathcal{M}$.

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[1] If the condition (1.2) is satisfied.

[2] That is, the metric which is induced on manifold $\mathcal{M}$ from the ambient space R^{n+p}.

Remark 1.2. In the case when $\mathcal{M}$ is compact without boundary and even dimensional, say $n = 2m$, the highest coefficient $w_{2m}(\mathcal{M})$ is especially interesting:

$$w_{2m}(\mathcal{M}) = (2\pi)^m \chi(\mathcal{M}), \tag{1.9}$$

where $\chi(\mathcal{M})$ is the Euler characteristic of $\mathcal{M}$. (See [Gra, Section 1.1].)

Definition 1.3. Let $\mathcal{M}$, $\dim \mathcal{M} = n$, be a smooth manifold, with or without boundary, $\mathcal{M} \subset \mathbb{R}^{n+p}$, $p \geq 1$. Let $\mathfrak{T}^{\mathbb{R}^{n+p}}_{\mathcal{M}}(t)$ be the tube of radius t around $\mathcal{M}$, see (1.4).

The polynomial $W^p_{\mathcal{M}}(t)$, which appears in the expression (1.5) :

$$\mathrm{Vol}_{n+p}\,(\mathfrak{T}^{\mathbb{R}^{n+p}}_{\mathcal{M}}(t)) = \omega_p t^p \cdot W^p_{\mathcal{M}}(t) \quad \text{for small positive } t, \tag{1.10}$$

is said to be the H. Weyl polynomial with the index p for the manifold $\mathcal{M}$.

Remark 1.4. The radius t of the tube is a positive number, so the formula (1.10) is meaningful for positive t only. However the polynomial $W^p_{\mathcal{M}}$ is determined uniquely by its restriction on any fixed interval $[0, \varepsilon]$, $\varepsilon > 0$, and we may and will consider this polynomial for every complex t.

Definition 1.5. Let $\mathcal{M}$ be a smooth manifold, $\dim \mathcal{M} = n$, which is embedded in the Euclidean space $\mathbb{R}^{n+p}$, $p \geq 1$, and let $W^p_{\mathcal{M}}$ be the Weyl polynomial of $\mathcal{M}$ (defined by (1.2), (1.10)). The coefficients $w_{2l}(\mathcal{M})$, $0 \leq l \leq [n/2]$, which are defined in terms of the Weyl polynomial $W^p_{\mathcal{M}}$ by the equality

$$W^p_{\mathcal{M}}(t) \overset{\text{def}}{=} \sum_{l=0}^{[\frac{n}{2}]} \frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\, w_{2l}(\mathcal{M}) \cdot t^{2l}\,, \tag{1.11}$$

are said to be the Weyl coefficients of the manifold $\mathcal{M}$.

Remark 1.6. Often, the factor in (1.11) appears in a expanded form:

$$\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)} = \frac{1}{(p+2)(p+4)\,\cdots\,(p+2l)}\,. \tag{1.12}$$

Remark 1.7. In defining the Weyl polynomials $W^p_{\mathcal{M}}$ of the manifold $\mathcal{M}$ by (1.10), we assumed that $\mathcal{M}$ is already embedded into $\mathbb{R}^{n+p}$. The tube around $\mathcal{M}$ and its volume are of primary importance in this definition, so that we, in fact, define the notion of the Weyl polynomial not for the manifold $\mathcal{M}$ itself but for manifold $\mathcal{M}$, which is already embedded in an ambient space. Moreover, we assume implicitly that from the outset the manifold $\mathcal{M}$ carries a 'natural' Riemannian metric and that this 'original' Riemannian metric coincides with the metric on $\mathcal{M}$ induced by the ambient space $\mathbb{R}^{n+p}$. (In other words, we assume that the imbedding is isometrical.) However, in this approach the 'original' metric does not play an 'explicit' role in the definition (1.1)-(1.10)-(1.11) of the Weyl polynomial $W^p_{\mathcal{M}}$ and the Weyl coefficients $w_{2l}(\mathcal{M})$.

There is another approach to defining the Weyl coefficients and the Weyl polynomials which does not require an actual embedding $\mathcal{M}$ into the ambient

space. Starting from the given Riemannian metric on $\mathcal{M}$, the Weyl coefficients $w_{2l}(\mathcal{M})$ can be introduced formally, by means of the Hermann Weyl expressions for $w_{2l}(\mathcal{M})$ in terms of the given metric on $\mathcal{M}$. Then the Weyl polynomials $W^{p}_{\mathcal{M}}(t)$ can be defined by means of the expression (1.11). In this approach, the intrinsic metric of $\mathcal{M}$ is of primary importance, but not the tubes around $\mathcal{M}$ and their volumes.

If the codimension p of $\mathcal{M}$ equals one[3] and $\dim \mathcal{M} = n$, the Weyl polynomial is of the form:

$$\mathrm{Vol}_{n+1}(\mathfrak{T}^{\mathbb{R}^{n+1}}_{\mathcal{M}}(t)) = 2t \cdot W^1_{\mathcal{M}}(t)\,, \quad W^1_{\mathcal{M}}(t) = \sum_{l=0}^{[\frac{n}{2}]} u_{2l}(\mathcal{M}) \cdot t^{2l}, \tag{1.13}$$

where

$$u_{2l}(\mathcal{M}) = \frac{2^{-l}\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+l+1)}\, w_{2l}(\mathcal{M})\,, \quad 0 \le l \le [\tfrac{n}{2}]\,. \tag{1.14}$$

In (1.13) the 'shortened' notation is used: $u_{2l}(\mathcal{M})$ instead of $u_{2l,1}(\mathcal{M})$. The factor $2t$ is the one-dimensional volume of the one-dimensional ball of radius t, that is, the length of the interval $[-t, t]$.

If the *hypersurface* $\mathcal{M}$ is orientable[4], then the tube $\mathfrak{T}_{\mathcal{M}}(t)$ can be decomposed into the union of two *half-tubes*, say, $\mathfrak{T}^+_{\mathcal{M}}(t)$ and $\mathfrak{T}^-_{\mathcal{M}}(t)$. The half-tubes $\mathfrak{T}^+_{\mathcal{M}}(t)$ and $\mathfrak{T}^-_{\mathcal{M}}(t)$ are the parts of the tube $\mathfrak{T}_{\mathcal{M}}(t)$ which are situated on the distinct sides of $\mathcal{M}$. In particular, if the hypersurface $\mathcal{M}$ is the boundary of a set $V : \mathcal{M} = \partial V$, then

$$\mathfrak{T}^+_{\mathcal{M}}(t) = \mathfrak{T}_{\mathcal{M}}(t) \setminus V, \quad \mathfrak{T}^-_{\mathcal{M}}(t) = \mathfrak{T}_{\mathcal{M}}(t) \cap V\,. \tag{1.15}$$

The $(n+1)$- dimensional volumes $\mathrm{Vol}_{n+1}(\mathfrak{T}^+_{\mathcal{M}}(t))$ and $\mathrm{Vol}_{n+1}(\mathfrak{T}^-_{\mathcal{M}}(t))$ of the half-tubes are also polynomials of t. These polynomials are of the form[5]:

$$\mathrm{Vol}_{n+1}(\mathfrak{T}^+_{\mathcal{M}}(t)) = t\, W^+_{\mathcal{M}}(t), \quad \mathrm{Vol}_{n+1}(\mathfrak{T}^-_{\mathcal{M}}(t)) = t\, W^-_{\mathcal{M}}(t)\,, \tag{1.16}$$

where:

$$W^+_{\mathcal{M}}(t) = \sum_{l=0}^{[\frac{n}{2}]} u_{2l}(\mathcal{M}) \cdot t^{2l} + t \sum_{l=0}^{[\frac{n+1}{2}]-1} u_{2l+1}(\mathcal{M}) \cdot t^{2l}, \tag{1.17a}$$

$$W^-_{\mathcal{M}}(t) = \sum_{l=0}^{[\frac{n}{2}]} u_{2l}(\mathcal{M}) \cdot t^{2l} - t \sum_{l=0}^{[\frac{n+1}{2}]-1} u_{2l+1}(\mathcal{M}) \cdot t^{2l}, \tag{1.17b}$$

and the coefficients $u_{2l}(\mathcal{M})$ are the same as those in (1.13)-(1.14). Unlike the coefficients $u_{2l}(\mathcal{M})$, the coefficients $u_{2l+1}(\mathcal{M})$ depend not only on the 'intrinsic' metric

[3] In other words, $\mathcal{M}$ is a hypersurface in $\mathbb{R}^{n+1}$.

[4] The orientation of the hypersurface $\mathcal{M}$ can be specified by means of the continuous vector field of unit normals on $\mathcal{M}$. The half-tubes $\mathfrak{T}^+_{\mathcal{M}}(t)$ and $\mathfrak{T}^-_{\mathcal{M}}(t)$ are the parts of the tube $\mathfrak{T}_{\mathcal{M}}(t)$ corresponding to the 'positive' and 'negative', respectively, directions of these normals.

[5] The equalities (1.16), (1.17) are some of the results of the theory of tubes around manifolds. See [Gra], [BeGo],[AdTa]

of the manifold $\mathcal{M}$, but also on how $\mathcal{M}$ is embedded into $\mathbb{R}^{n+1}$. It is remarkable that when the volumes of the half-tubes are summed:

$$2\,W_{\mathcal{M}}(t) = W_{\mathcal{M}}^{+}(t) + W_{\mathcal{M}}^{-}(t),$$

the dependence on how $\mathcal{M}$ is embedded disappears. As it is seen from (1.17), $W_{\mathcal{M}}^{-}(t) = W_{\mathcal{M}}^{+}(-t)$, hence

$$2\,W_{\mathcal{M}}(t) = W_{\mathcal{M}}^{+}(t) + W_{\mathcal{M}}^{+}(-t). \tag{1.18}$$

We remark also that the volumes of the half-tubes can be expressed only in the terms of the polynomial $W_{\mathcal{M}}^{+}$:

$$\mathrm{Vol}_{n+1}(\mathfrak{T}_{\mathcal{M}}^{+}(t)) = t\,W_{\mathcal{M}}^{+}(\ t\ ) \quad \text{for small positive } t\,. \tag{1.19a}$$

$$\mathrm{Vol}_{n+1}(\mathfrak{T}_{\mathcal{M}}^{-}(t)) = t\,W_{\mathcal{M}}^{+}(-t) \quad \text{for small positive } t\,. \tag{1.19b}$$

The theory of the tubes around manifolds is presented in [Gra], and to some extent in [BeGo], Chapter 6, and in [AdTa], Chapter 10. The comments of V.Arnold [Arn] to the Russian translations of the paper [Wey1] by H.Weyl are very rich in content.

In the event that the hypersurface $\mathcal{M}$ is the boundary of a convex set V: $\mathcal{M} = \partial V$, the Weyl polynomial $W_{\mathcal{M}}^{1}$ can be expressed in terms of polynomials considered in the theory of convex sets.

In the theory of convex sets the following fact, which was discovered by Hermann Minkowski, [Min1, Min2], is of principal importance: *Let V_1 and V_2 be compact convex sets in $\mathbb{R}^n$. For positive numbers t_1, t_2, let us form the 'linear combination' $t_1V_1 + t_2V_2$ of the sets V_1 and V_2. (That is, $t_1V_1 + t_2V_2 = \{t_1x_1 + t_2x_2 : x_1 \in V_1,\, x_2 \in V_2\}$.) Then the n-dimensional Euclidean volume $\mathrm{Vol}_n(t_1V_1 + t_2V_2)$ of this linear combination, considered as a function of the variables t_1, t_2, is a homogeneous polynomial of degree n. (It may vanish identically.)*

Choosing V as V_1 and the unit ball B^n of $\mathbb{R}^n$ as V_2, we obtain the following:

Let V be a compact convex set in $\mathbb{R}^n$, B^n be the unit ball in $\mathbb{R}^n$. Then the n-dimensional volume $\mathrm{Vol}_n(V + tB^n)$, considered as a function of the variable $t \in [0,\infty)$, is a polynomial of degree n.

Definition 1.8. Let $V \subset \mathbb{R}^n$ be a compact convex set. The polynomial which expresses the n-dimensional volume of the linear combination $V + tB^n$ as a function of the variable $t \in [0,\infty)$ is said to be the Steiner polynomial of the set V and is denoted by $S_V^{\mathbb{R}^n}(t)$:

$$S_V^{\mathbb{R}^n}(t) = \mathrm{Vol}_n(V + tB^n)\,, \quad (t \in [0,\infty)). \tag{1.20}$$

The coefficients of a Steiner polynomial are denoted by $s_k^{\mathbb{R}^n}(V)$:

$$S_V^{\mathbb{R}^n}(t) = \sum_{0\le k\le n} s_k^{\mathbb{R}^n}(V)t^k. \tag{1.21}$$

If there is no need to emphasize that the ambient space is $\mathbb{R}^n$, then the shortened notation $S_V(t)$, $s_k(V)$ for the Steiner polynomial and its coefficients,respectively, will be used.

Of course,

$$S_V^{\mathbb{R}^n}(t) = \mathrm{Vol}_n(\mathfrak{V}_V^{\mathbb{R}^n}(t)),$$

where $\mathfrak{V}_V^{\mathbb{R}^n}(t)$ is the t-neighborhood of the set V with respect to $\mathbb{R}^n$:

$$\mathfrak{V}_V^{\mathbb{R}^n}(t) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, V) \leq t\}. \tag{1.22}$$

It is evident that

$$s_0(V) = \mathrm{Vol}_n(V), \quad \text{and} \quad s_n(V) = \mathrm{Vol}_n(B^n). \tag{1.23}$$

If the boundary ∂V of a convex set V is smooth, then the $(n-1)$-dimensional volume ('the area') of the boundary ∂V can be expressed as

$$s_1(V) = \mathrm{Vol}_{n-1}(\partial V). \tag{1.24}$$

For a convex set V, whose boundary ∂V may be non-smooth, the formula (1.24) serves as a *definition* of the *'area'* of ∂V. (See [BoFe, **31**], [Min1, § 24], [Web, **6.4**].) Let us emphasize that the Steiner polynomial is defined for an *arbitrary* compact convex set V, without any extra assumptions. The boundary of V may be non-smooth, and the interior of V may be empty. In particular, the Steiner polynomial is defined for any convex polytope.

Definition 1.9. Let $V \subset \mathbb{R}^n$ be a convex set. V is said to be solid if the interior of V is not empty, and non-solid if the interior of V is empty.

Definition 1.10. A set $\mathcal{M} \subset \mathbb{R}^{n+1}$ is called an n-dimensional closed convex surface if there exists a solid compact convex set $V \subset \mathbb{R}^{n+1}$, such that

$$\mathcal{M} = \partial V. \tag{1.25}$$

The set V is said to be the generating set for the surface $\mathcal{M}$.

Lemma 1.11. *If the closed n - dimensional convex surface $\mathcal{M}$ is also a smooth manifold, then the Weyl polynomial $W^1_{\mathcal{M}}$ of the surface $\mathcal{M}$ and the Steiner polynomial $S_V^{\mathbb{R}^{n+1}}$ of its generating set V are related in the following way:*

$$2t\, W^1_{\mathcal{M}}(t) = S_V^{\mathbb{R}^{n+1}}(t) - S_V^{\mathbb{R}^{n+1}}(-t). \tag{1.26}$$

Proof of Lemma 1.11. We assign the positive orientation to the vector field of exterior normals on ∂V. Let $\mathfrak{T}^+_{\partial V}(t)$ be the 'exterior' half-tube around ∂V. For positive t,

$$V + tB^{n+1} = V \cup \mathfrak{T}^+_{\partial V}(t),$$

Moreover the set V and $\mathfrak{T}^+_{\partial V}(t)$ do not intersect. Therefore,

$$\mathrm{Vol}_{n+1}(V + tB^{n+1}) = \mathrm{Vol}_{n+1}(V) + \mathrm{Vol}_{n+1}(\mathfrak{T}^+_{\partial V}(t)).$$

Hence,

$$S_V(t) = S_V(0) + t\, W^+_{\mathcal{M}}(t), \quad \mathcal{M} = \partial V,$$

where $W^+_{\mathcal{M}}$ is a polynomial of the form (1.16) (with n replaced by $n+1$: now $\dim V = n+1$). It follows, furthermore, that:

$$S_V(-t) = S_V(0) - t\,W^+_{\mathcal{M}}(-t).$$

The equality (1.26) follows from the latter equality and from (1.18). □

Since the Steiner polynomial is defined for an arbitrary compact convex set, the formula (1.26) can serve as a *definition* for the Weyl polynomial of an *arbitrary* closed convex surface, smooth or non-smooth. Furthermore, we can define the Weyl polynomial for the 'improper convex surface ∂V', where V is a non-solid compact convex set.

Definition 1.12. Let V, $V \subset \mathbb{R}^{n+1}$, be a compact convex set. The boundary ∂V of the set V is said to be the boundary surface of V. The boundary surface of V is said to be proper if V is solid, and improper if V is non-solid.

The following improper closed convex surface plays a role in what follow:

Definition 1.13. Let $V \subset \mathbb{R}^n$ be a compact convex set, which is solid *with respect to* $\mathbb{R}^n$. We identify $\mathbb{R}^n$ with its image $\mathbb{R}^n \times 0$ using the 'canonical' embedding[6] $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ and the set V with the set $V \times 0$, considered as a subset of $\mathbb{R}^{n+1}$: $V \times 0 \subset \mathbb{R}^{n+1}$. The set $V \times 0$, considered as a subset of $\mathbb{R}^{n+1}$, is said to be the squeezed cylinder with the base V.

Remark 1.14. The set $V \times 0$ can be interpreted as a ' cylinder of height' zero, whose 'lateral surface' is the Cartesian product $\partial V \times [0,0]$ and whose bases, lower and upper, are given the sets $V \times (-0)$ and $V \times (+0)$, respectively:

$$\partial(V \times 0) = \big((\partial V) \times [0,0]\big) \cup \big(V \times (-0)\big) \cup \big(V \times (+0)\big)\,. \tag{1.27}$$

In other words, the boundary surface $\partial(V \times 0)$ can be considered as 'the doubly covered' set V. In particular,

$$\dim \partial(V \times 0) = n\,. \tag{1.28}$$

and the number $\mathrm{Vol}_n(V \times (-0)) + \mathrm{Vol}_n(V \times (+0)) = 2\,\mathrm{Vol}_n(V)$ can be naturally interpreted as the 'n- dimensional area' of the n- dimensional convex surface (improper) $\partial(V \times 0)$:

$$\mathrm{Vol}_n(\partial(V \times 0)) = 2\,\mathrm{Vol}_n(V)\,. \tag{1.29}$$

On the other hand, the equality (1.24), in which the squeezed cylinder $V \times 0 \subset \mathbb{R}^{n+1}$ plays the role of the set $V \subset \mathbb{R}^n$, takes the form

$$\mathrm{Vol}_n(\partial(V \times 0)) = s_1^{\mathbb{R}^{n+1}}(V \times 0)\,, \tag{1.30}$$

where $s_k^{\mathbb{R}^{n+1}}(V \times 0)$, $k = 0, 1, \ldots, n+1$, are the coefficients of the Steiner polynomial $S^{\mathbb{R}^{n+1}}_{V \times 0}(t)$ of the squeezed cylinder $V \times 0$ with respect to the ambient space $\mathbb{R}^{n+1}$. (See (1.21).)

[6] The point $x \in \mathbb{R}^n$ is identified with the point $(x, 0) \in \mathbb{R}^{n+1}$.

In section 11 we prove the following statement, which there appears as Lemma 11.2:

Lemma 1.15. *Let V be a compact convex set in $\mathbb{R}^n$, and*

$$S_V^{\mathbb{R}^n}(t) = \sum_{0\le k\le n} s_k^{\mathbb{R}^n}(V)t^k \tag{1.31}$$

be the Steiner polynomial with respect to the ambient space $\mathbb{R}^n$. Then the Steiner polynomial $S_{V\times 0}^{\mathbb{R}^{n+1}}(t)$ with respect to the ambient space $\mathbb{R}^{n+1}$ is equal to:

$$S_{V\times 0}^{\mathbb{R}^{n+1}}(t) = t\sum_{0\le k\le n} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\, s_k^{\mathbb{R}^n}(V)\, t^k\,. \tag{1.32}$$

So,

$$s_0^{\mathbb{R}^{n+1}}(V\times 0) = 0, \quad s_{k+1}^{\mathbb{R}^{n+1}}(V\times 0) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\, s_k^{\mathbb{R}^n}(V),\ k=0,\,\dots,\,n\,.$$

In particular, $s_1^{\mathbb{R}^{n+1}}(V\times 0) = 2s_0^{\mathbb{R}^n}(V)$. Since $s_0^{\mathbb{R}^n}(V) = \mathrm{Vol}_n(V)$, see (1.23),

$$s_1^{\mathbb{R}^{n+1}}(V\times 0) = 2\,\mathrm{Vol}_n(V)\,. \tag{1.33}$$

The equalities (1.29), (1.30) and (1.33) agree.

Remark 1.16. Any non-solid compact convex set V can be presented as the limit (in the Hausdorff metric) of a monotonic[7] family $\{V_\varepsilon\}_{\varepsilon>0}$ of solid convex sets V_ε :

$$V = \lim_{\varepsilon\to+0} V_\varepsilon.$$

Moreover, the approximating family $\{V_\varepsilon\}_{\varepsilon>0}$ of convex sets can be chosen so that the boundary $\partial(V_\varepsilon)$ of each set V_ε is a smooth surface. Thus, the improper convex surface ∂V may be presented as the limit of proper convex smooth surfaces $\partial(V_\varepsilon)$ which shrink to ∂V:

$$\partial V = \lim_{\varepsilon\to+0} \partial(V_\varepsilon).$$

Definition 1.17. Let V, $V\subset\mathbb{R}^{n+1}$, be an arbitrary compact convex set. The Weyl polynomial $W_{\partial V}^1(t)$ of the convex surface $\mathcal{M}=\partial V$, proper or improper, is defined by the formula (1.26). In other words, the Weyl polynomial $t\,W_{\partial V}^1$ is defined as the odd part of the Steiner polynomial $S_V^{\mathbb{R}^{n+1}}$:

$$t\cdot W_{\partial V}^1(t) = {}^{\mathcal{O}}\!S_V^{\mathbb{R}^{n+1}}(t), \tag{1.34}$$

where the even part ${}^{\mathcal{E}}\!P$ and the odd part ${}^{\mathcal{O}}\!P$ of an arbitrary polynomial P are defined as ${}^{\mathcal{E}}\!P(t)=\frac{1}{2}(P(t)+P(-t))$, ${}^{\mathcal{O}}\!P(t)=\frac{1}{2}(P(t)-P(-t))$, respectively. (See Definition 7.2.)

[7]The monotonicity means that $V_{\varepsilon'}\supset V_{\varepsilon''}\supset V$ for $\varepsilon'>\varepsilon''>0$.

Remark 1.18. In the case when the set V is solid and its boundary ∂V is smooth, both definitions, Definition 1.17 and Definition 1.3 of the Weyl polynomial $W^1_{\partial V}$, are applicable to ∂V. In this case both definitions agree.

Remark 1.19. Why would it be useful to consider improper convex surfaces and their Weyl polynomials?
As it was mentioned earlier, (Remark 1.16), every improper convex surface ∂V is a limiting object for a family of proper smooth convex surfaces $\partial(V_\varepsilon)$. It turns out that the Weyl polynomial for this improper surface is the limit of the Weyl polynomials for this 'approximating' family $\{V_\varepsilon\}_{\varepsilon>0}$ of smooth proper surfaces.
The Weyl polynomials for the improper surface ∂V may, therefore, be useful for studying the limiting behavior of the family of the Weyl polynomials for the proper surfaces $\partial(V_\varepsilon)$ shrinking to the improper surface ∂V. In particular, see Theorem 2.12 formulated at the end of Section 2, and its proof, presented at the end of Section 11.

Let $\mathcal{M}$ be an n-dimensional closed convex surface, which is not assumed to be smooth, and V the generating convex set for $\mathcal{M}$: $\mathcal{M} = \partial V$. Let $S_V^{\mathbb{R}^{n+1}}$ be the Steiner polynomial for V, defined by Definition 1.8. According to Definition 1.17, the Weyl polynomial $W^1_{\mathcal{M}}$ is equal to

$$W^1_{\mathcal{M}}(t) = \sum_{0\leq l\leq\left[\frac{n}{2}\right]} s_{2l+1}(V)t^{2l}, \tag{1.35}$$

or, alternatively,

$$u_{2l}(\mathcal{M}) = s_{2l+1}(V), \quad 0 \leq l \leq [\tfrac{n}{2}], \tag{1.36}$$

where $u_{2l}(\mathcal{M})$ are the coefficients of the Weyl polynomial $W^1_{\mathcal{M}}$, (1.13), of the n-dimensional surface $\mathcal{M}$ with respect to the ambient space $\mathbb{R}^{n+1}$ and $s_k(V)$, $k = 2l+1$, are the coefficients of the Steiner polynomial $S_V^{\mathbb{R}^{n+1}}$:

$$S_V^{\mathbb{R}^{n+1}}(t) = \mathrm{Vol}_{n+1}(V + tB^{n+1}), \quad S_V^{\mathbb{R}^{n+1}}(t) = \sum_{0\leq k\leq n+1} s_k(V)t^k\,. \tag{1.37}$$

Definition 1.20. Given a closed n-dimensional convex surface $\mathcal{M}$, proper or not, $\mathcal{M} = \partial V$, the numbers $w_{2l}(\mathcal{M})$, $0 \leq l \leq [\frac{n}{2}]$, are defined as

$$w_{2l}(\mathcal{M}) = 2^l \frac{\Gamma(l+\frac{1}{2}+1)}{\Gamma(\frac{1}{2}+1)}\, s^{\mathbb{R}^{n+1}}_{2l+1}(V), \tag{1.38}$$

where $s_k^{\mathbb{R}^{n+1}}(V)$, $k = 2l+1$, are the coefficients of the Steiner polynomial $S_V^{\mathbb{R}^{n+1}}$ for the generating set V, (1.37). The numbers $w_{2l}(\partial V)$, $0 \leq l \leq \left[\frac{n}{2}\right]$, are said to be the Weyl coefficients for the surface $\mathcal{M}$.

Remark 1.21. According to Lemma 1.15, in the event that the (improper) convex surface $\mathcal{M}, \dim\mathcal{M} = n$, is the boundary of the squeezed cylinder (see Definition 1.13), that is, if $\mathcal{M} = \partial(V \times 0)$, where $V \subset \mathbb{R}^n$, then the Weyl coefficients

$w_{2l}(\mathcal{M}),\ 0 \le l \le [\frac{n}{2}]$, are:

$$w_{2l}(\mathcal{M}) = 2^{l+1}\,\Gamma(l+1)\, s_{2l}^{\mathbb{R}^n}(V)\,, \tag{1.39}$$

where $s_k^{\mathbb{R}^n}(V)$, $k = 2l$, are the coefficients of the Steiner polynomial $S_V^{\mathbb{R}^n}$ for the base V of the squeezed cylinder $\partial(V \times 0)$.

Remark 1.22. In the case when the convex surface $\mathcal{M}$, $\mathcal{M} = \partial V$, is smooth and 'proper', that is, the set V generating the surface $\mathcal{M}$ is solid, both definitions, Definition 1.20 and Definition 1.5 of the Weyl coefficients $w_{2l}(\mathcal{M})$ are applicable. In this case, accordingly to (1.13)-(1.14) and (1.36)-(1.38),[8] both definitions agree.

Note, that according to (1.24), (see also Remark 1.14),

$$w_0(\mathcal{M}) = \mathrm{Vol}_n(\mathcal{M}) \tag{1.40}$$

for every n-dimensional closed convex surface $\mathcal{M}$.

Lemma 1.23. I. *Let V, $V \subset \mathbb{R}^n$, be a solid (with respect to $\mathbb{R}^n$) compact convex set. Then the coefficients $s_k^{\mathbb{R}^n}(V)$, $0 \le k \le n$, of its Steiner polynomials*[9] *are strictly positive: $s_k^{\mathbb{R}^n}(V) > 0,\ 0 \le k \le n$.*

II. *Let $\mathcal{M}$ be a proper compact convex surface,* $\dim \mathcal{M} = n$. *Then all its Weyl coefficients $w_{2l}(\mathcal{M})$ are strictly positive: $w_{2l}(\mathcal{M}) > 0,\ 0 \le l \le [\frac{n}{2}]$.*

III. *Let $\mathcal{M}$ be the boundary surface*[10] *of a squeezed cylinder, whose base V, $\dim V = n$, is a compact convex set which is solid with respect to $\mathbb{R}^n$. Then all its Weyl coefficients $w_{2l}(\mathcal{M})$ are strictly positive: $w_{2l}(\mathcal{M}) > 0,\ 0 \le l \le [\frac{n}{2}]$.*

Statement I of Lemma 1.23 is a consequence of a more general statement related to the monotonicity properties of the mixed volumes. This will be discussed later, in Section 8. Statements II and III of Lemma 1.23 are consequences of the statement I and (1.38), (1.39).

Definition 1.24. Given a closed n-dimensional convex surface $\mathcal{M}$, the Weyl polynomial $W_{\mathcal{M}}^p$ for $\mathcal{M}$ with the index p, $p = 1, 2, 3, \dots$, is defined as

$$W_{\mathcal{M}}^p(t) = \sum_{l=0}^{[\frac{n}{2}]} \frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)} w_{2l}(\mathcal{M}) \cdot t^{2l}\,, \tag{1.41}$$

where the Weyl coefficients $w_{2l}(\mathcal{M})$ are introduced in Definition 1.20.

Let us emphasize that in Definition 1.24 no assumption concerning the smoothness of the surface $\mathcal{M}$ are made. We already mentioned that the definitions of the Weyl coefficients w_{2l} for smooth manifolds and for convex surfaces agree. Therefore, if the convex surface $\mathcal{M}$ is also a smooth manifold, then the definitions 1.3 and 1.24 of the Weyl polynomial agree as well.

We also define the infinite index Weyl polynomial $W_{\mathcal{M}}^{\infty}$.

[8] Actually, the equalities (1.14), (1.36) served as a motivation for Definition 1.20.
[9] See (1.20), (1.21).
[10] See Definition 1.13 and Remark 1.14.

Definition 1.25. Let $\mathcal{M}$, $\dim\mathcal{M} = n$ be either a smooth manifold, or a closed compact convex surface, and let $w_{2l}(\mathcal{M})$, $l = 0, 1, \dots, [\frac{n}{2}]$, be the Weyl coefficients of $\mathcal{M}$, defined by Definition 1.5 in the smooth case, and by Definition 1.20 in the convex case. The infinite index Weyl polynomial $W_{\mathcal{M}}^{\infty}$ is defined as

$$W_{\mathcal{M}}^{\infty}(t) = \sum_{l=0}^{[\frac{n}{2}]} w_{2l}(\mathcal{M}) \cdot t^{2l}. \tag{1.42}$$

Remark 1.26. In view of (1.12),

$$W_{\mathcal{M}}^{p}(\sqrt{p}t) = w_0(\mathcal{M}) + \sum_{l=1}^{[\frac{n}{2}]} \frac{p^l}{(p+2)(p+4)\,\cdots\,(p+2l)}\, w_{2l}(\mathcal{M}) \cdot t^{2l}\,.$$

Therefore, the polynomial $W_{\mathcal{M}}^{\infty}(t)$ can be considered as a limiting object for the family $\bigl\{W_{\mathcal{M}}^{p}(\sqrt{p}t)\bigr\}_{p=1,\,2,\,3,\,\dots}$ of the (renormalized) Weyl polynomials of the index p:

$$W_{\mathcal{M}}^{\infty}(t) = \lim_{p\to\infty} W_{\mathcal{M}}^{p}(\sqrt{p}t)\,. \tag{1.43}$$

Thus, the sequence $\bigl\{W_{\mathcal{M}}^{p}\bigr\}_{p=1,\,2,\,3,\,\dots}$ of the Weyl polynomials, $\deg W_{\mathcal{M}}^{p} = 2\bigl[\frac{n}{2}\bigr]$, as well as the 'limiting' polynomial $W_{\mathcal{M}}^{\infty}$, are associated with any closed n-dimensional convex surface $\mathcal{M}$.

Weyl polynomials (and Steiner polynomials in the convex case) somehow describe intrinsic properties of the appropriate manifolds. On the other hand, there are remarkable geometrical objects such as regular polytopes, compact matrix groups, spaces of constant curvatures, etc. Our belief is that the Weyl polynomials associated with these geometric objects are of fundamental importance and possess interesting properties. These polynomials should be carefully studied. In particular, the following question is natural:

What can we say about the roots of such polynomials?

Remark 1.27. In the theory of lattice polytopes, the *Ehrhart polynomials* are a counterpart to the Steiner polynomials. For more on the Ehrhart polynomials we refer to [BeRo]. See also [Gru]. Location of the roots of Ehrhart polynomials was studied in [BLD], [BHW].

2. Formulation of main results.

In this section we formulate the main results on the locations of roots belongings to Steiner and Weyl polynomials related to convex sets and surfaces.

Dissipative and conservative polynomials. We introduce two classes of polynomials: dissipative polynomials and conservative polynomials. In many cases the Steiner polynomials related to convex sets are dissipative and the Weyl polynomials are conservative.

Definition 2.1. The polynomial M is said to be dissipative if all roots of M are situated in the open left half plane $\{z : \operatorname{Re} z < 0\}$. The dissipative polynomials are also called the Hurwitz polynomials or the stable polynomials.

Definition 2.2. The polynomial W is said to be conservative if all roots of W are purely imaginary and simple, in other words,if all roots of W are contained in the imaginary axis $\{z : \operatorname{Re} z = 0\}$ and each of them is of multiplicity one.

Theorem 2.3. *Given a closed compact convex surface $\mathcal{M}$, $\dim \mathcal{M} = n$, $\mathcal{M} = \partial V$, let $W^1_{\mathcal{M}}$ be the Weyl polynomial of index 1 associated with $\mathcal{M}$, and let $S^{\mathbb{R}^{n+1}}_V$ be the Steiner polynomial associated with the set V.*

If the polynomial $S^{\mathbb{R}^{n+1}}_V$ is dissipative, then the polynomial $W^1_{\mathcal{M}}$ is conservative.

The proof of Theorem 2.3 is based on the relation (1.26). Theorem 2.3 is derived from (1.26) using the Hermite-Biehler Theorem. We do this in Section 7.

From (1.43) it follows that if for every p the polynomial $W^p_{\mathcal{M}}$ has only purely imaginary roots, then all the roots of the polynomial $W^\infty_{\mathcal{M}}$ are purely imaginary as well. In particular, *all the roots of the polynomial $W^\infty_{\mathcal{M}}$ are purely imaginary if for every p the polynomial $W^p_{\mathcal{M}}$ is conservative.*

However, what is important for us is that, the converse statement:

Lemma 2.4. *If the polynomial $W^\infty_{\mathcal{M}}$ is conservative, then all the polynomials $W^p_{\mathcal{M}}$, $p = 1, 2, 3, \dots$, are conservative as well.*

Lemma 2.4 is the consequence of a result of Laguerre about the multiplier sequences. Proof of Lemma 2.4 appears at the end of Section 6.

Keeping Lemma 2.4 in mind, we will concentrate our efforts on the study of the location of the roots of the Weyl polynomial $W^\infty_{\mathcal{M}}$ with infinite index.

The case of low dimension. In this section we discuss the Steiner polynomials of convex sets V, $V \subset \mathbb{R}^n$, and the Weyl polynomials of closed convex surfaces $\mathcal{M}$, $\dim \mathcal{M} = n$, for 'small' n: $n = 2, 3, 4, 5$.

Theorem 2.5. *Let n be one of the numbers 2, 3, 4 or 5, and let V, $V \subset \mathbb{R}^n$, be a solid compact convex set. Then the Steiner polynomial $S^{\mathbb{R}^n}_V$ is dissipative.*

Theorem 2.6. *Let n be one of the numbers 2, 3, 4 or 5, and let $\mathcal{M}$ be a closed proper*[11] *convex surface of dimension n.*

Then the following hold:

1. *The Weyl polynomial $W^\infty_{\mathcal{M}}$ with infinite index is conservative.*

[11] That is, the generating set V is solid.

2. *For every $p = 1, 2, 3, \dots$, the Weyl polynomial $W^p_{\mathcal{M}}$ with index p is conservative.*

Remark 2.7. After this work was completed, Martin Henk called our attention to Remark 4.4 of [Tei], which appears on the last page of this paper. In this remark, the statement is formulated which is very close to our Theorem 2.5. Some negative results are stated there as well. Detailed proofs are lacking.

Theorem 2.5 and 2.6 are proved in Section 10. In proving these theorems, we combine the Routh-Hurwitz Criterion, which expresses the property of a polynomial to be dissipative in terms of its coefficients, and the Alexandrov-Fenchel inequalities, which express the logarithmic convexity property for the sequence of the cross-sectional measures of a convex set.

Selected 'regular' convex sets: balls, cubes, squeezed cylinders. For large n, the statements analogous to Theorems 2.5 and 2.6 do not hold. If n is large enough, then there exists solid compact convex sets[12] V, $\dim V = n$, such that the Minkowski polynomials $S_V^{\mathbb{R}^{n+1}}$ are not dissipative and the Weyl polynomials $W^p_{\partial V}$ are not conservative. However, for some 'regular' convex sets V, like balls and cubes, the Weyl polynomials $W^p_{\partial V}$ are conservative and the Steiner polynomials are dissipative in any dimension.

Let us present the collection of 'regular' convex sets and their boundary surfaces, which we will be dealing with further on. Such sets and surfaces will be considered for every n, so that we, in fact, consider families of sets (indexed by their dimensions) and not single sets.

◊ The unit ball B^n:

$$B^n = \{x = (x_1, \dots , x_n) \in \mathbb{R}^n : \sum_{1\le k\le n} |x_k|^2 \le 1\}, \tag{2.1}$$

$$\mathrm{Vol}_n(B^n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\,. \tag{2.2}$$

◊ The squeezed spherical cylinder $B^n \times 0$, $B^n \times 0 \subset \mathbb{R}^{n+1}$.

◊ The unit sphere,

$$S^n = \{x = (x_1, \dots , x_n, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{1\le k\le n+1} |x_k|^2 = 1\},$$

in other words, the boundary surface of the unit ball: $S^n = \partial B^{n+1}$,

$$\mathrm{Vol}_n(S^n) = (n+1)\,\mathrm{Vol}_{n+1}(B^{n+1})\,. \tag{2.3}$$

◊ The boundary surface of the squeezed spherical cylinder $\partial(B^n \times 0)$:

$$\mathrm{Vol}_n(\partial(B^n \times 0)) = 2\,\mathrm{Vol}_n(B^n)\,. \tag{2.4}$$

[12] Very flat ellipsoids can be taken as such V. See Theorem 2.12.

◊ The unit cube Q^n:

$$Q^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \max_{1\le k\le n} |x_k| \le 1\}, \tag{2.5}$$

$$\mathrm{Vol}_n(Q^n) = 2^n\,. \tag{2.6}$$

◊ The squeezed cubic cylinder $Q^n \times 0$, $Q^n \times 0 \subset \mathbb{R}^{n+1}$.

◊ The boundary surface ∂Q^{n+1} of the unit cube:

$$\mathrm{Vol}_n(\partial Q^{n+1}) = (n+1)\,\mathrm{Vol}_{n+1}(Q^{n+1}). \tag{2.7}$$

◊ The boundary surface of the squeezed cubic cylinder $\partial(Q^n \times 0)$:

$$\mathrm{Vol}_n(\partial(Q^n \times 0)) = 2\,\mathrm{Vol}_n(Q^n)\,. \tag{2.8}$$

Locating roots of the Steiner and Weyl polynomials related to 'regular' convex sets.

Let us state the main results about locating roots of the Steiner polynomials and the Weyl polynomials related to the above mentioned 'regular' convex sets and their surfaces.

Theorem 2.8. *For every $n = 1, 2, 3, \dots$:*

1. *The Steiner polynomial $S^{\mathbb{R}^n}_{B^n}$ associated with the ball B^n is dissipative, moreover all its roots are negative*[13].
2. *The Steiner polynomial $S^{\mathbb{R}^{n+1}}_{B^n\times 0}$ associated with the squeezed spherical cylinder $B^n \times 0$ is of the form*[14] *$S^{\mathbb{R}^{n+1}}_{B^n\times 0}(t) = t\cdot D^{\mathbb{R}^{n+1}}_{B^n\times 0}(t)$, where the polynomial $D^{\mathbb{R}^{n+1}}_{B^n\times 0}$ is dissipative. If n is large enough, then the polynomial $S^{\mathbb{R}^{n+1}}_{B^n\times 0}$ has non-real roots.*
3. *The Steiner polynomial $S^{\mathbb{R}^n}_{Q^n}$ associated with the cube Q^n is dissipative, moreover all its roots are negative.*
4. *The Steiner polynomial $S^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ associated with the squeezed cubical cylinder $Q^n \times 0$ is of the form*[14] *$S^{\mathbb{R}^{n+1}}_{Q^n\times 0}(t) = t\cdot D^{\mathbb{R}^{n+1}}_{Q^n\times 0}(t)$, where the polynomial $D^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ is dissipative and all roots of the polynomial $D^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ are negative.*

Theorem 2.9. *For every $n = 1, 2, 3, \dots$:*

1. *The Weyl polynomials $W^{\infty}_{\partial B^{n+1}}(t)$ of infinite index, as well as the Weyl polynomials $W^{p}_{\partial B^{n+1}}(t)$ of arbitrary finite index p, $p = 1, 2, \dots$, associated with the boundary surface of the ball B^{n+1} are conservative.*
2. *The Weyl polynomials $W^{p}_{\partial(B^n\times 0)}$ of order*[15] *$p = 1$, $p = 2$ and $p = 4$ associated with the boundary surface of the squeezed spherical cylinder $B^n \times 0$ are conservative.*
3. *The Weyl polynomials $W^{\infty}_{\partial Q^{n+1}}(t)$ of infinite index, as well as the Weyl polynomials $W^{p}_{\partial Q^{n+1}}(t)$ of arbitrary finite index p, $p = 1, 2, \dots$, associated with the boundary surface of the cube Q^{n+1} are conservative.*

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[13] This part of the Theorem is trivial: $S^{\mathbb{R}^n}_{B^n}(t) = (1+t)^n$

[14] The factors t appears because the set $B^n \times 0$ is not solid in $\mathbb{R}^{n+1}$.

[15] The case $p = 3$ remains open.

4. *The Weyl polynomials* $W^{\infty}_{\partial(Q^n\times 0)}(t)$ *of infinite index, as well as the Weyl polynomials* $W^{p}_{\partial(Q^n\times 0)}(t)$ *of arbitrary finite index* p, $p=1,\,2,\,\dots\,,$ *associated with the boundary surface of the squeezed cubic cylinder* $Q^n\times 0$ *are conservative.*

Remark 2.10. The roots of the Weyl polynomial $W^1_{\partial B^{n+1}}$ can be found explicitly. Indeed

$$W^1_{\partial B^{n+1}}(it)=\mathrm{Vol}_{n+1}(B^{n+1})\frac{1}{2it}\big((1+it)^{n+1}-(1-it)^{n+1}\big)\,.$$

Changing variable

$$t\to\varphi:\ 1+it=|1+it|e^{i\varphi},t=\mathrm{tg}\,\varphi\,,\quad -\frac{\pi}{2}<\varphi<\frac{\pi}{2}\,,$$

we reduce the equation $W^1_{\partial B^{n+1}}(it)=0$ to the equation

$$\frac{\sin\,(n+1)\varphi}{\sin\varphi}=0,\quad -\frac{\pi}{2}<\varphi<\frac{\pi}{2}\,.$$

The roots of the latter equation are:

$$\varphi_k=\frac{k\pi}{n+1},\qquad -\Big[\frac{n}{2}\Big]\leq k\leq\Big[\frac{n}{2}\Big],\quad k\neq 0\,.$$

So, the roots t_k of the equation $W^1_{\partial B^{n+1}}(it)=0$ are

$$t_k=\mathrm{tg}\,\tfrac{k\pi}{n+1},\qquad -\big[\tfrac{n}{2}\big]\leq k\leq\big[\tfrac{n}{2}\big]\,,\quad k\neq 0\,.$$

In particular, the polynomial $W^1_{S^n}$ is conservative.

Negative results:

Theorem 2.11. *Let* $p\in\mathbb{Z}$ *with* $p\geq 5$. *If* n *is large enough:* $n\geq N(p)$, *then the Weyl polynomial* $W^p_{\partial(B^n\times 0)}$ *is not conservative: some of its roots do not belong to the imaginary axis.*

For an integer $q:q\geq 1$, let $E_{n,\,q,\,\varepsilon}$ be the $n+q$- dimensional ellipsoid:

$$E_{n,\,q,\,\varepsilon}=\{(x_1,\,x_2,\,\dots\,,\,x_n,\,\dots\,,\,x_{n+q})\in\mathbb{R}^{n+q}:\sum_{0\leq j\leq n+q}(x_j/a_j)^2\leq 1\},\tag{2.9a}$$

where

$$a_j=1\ \text{ for }\ 1\leq j\leq n,\quad a_j=\varepsilon\ \text{ for }\ n+1\leq j\leq n+q\,.\tag{2.9b}$$

Theorem 2.12.

1. *Let* $q\in\mathbb{Z}$ *with* $5\leq q<\infty$. *If* n *is large enough:* $n\geq N(q)$, *and* ε *is small enough:* $0<\varepsilon\leq\varepsilon(n,\,q)$, *then the Steiner polynomial* $S^{\mathbb{R}^{n+q}}_{E_{n,\,q,\,\varepsilon}}$ *is not dissipative: some of its roots are situated in the open right-half plane.*
2. *Let* $p,q\in\mathbb{Z}$ *such that* q *is odd,* $p\geq 1,\,q\geq 1,\,p+q\geq 6\,.$ *If* n *is large enough:* $n\geq N(p,q)$ *and* ε *is small enough:* $0<\varepsilon\leq\varepsilon(n,\,p,\,q)$, *then the Weyl polynomial* $W^p_{E_{n,\,q,\,\varepsilon}}$ *is not conservative: some of its roots do not belong to the imaginary axis.*

Proof of Theorem 2.12 is presented in Section 11.

Remark 2.13. In the recent paper [HeHe] other examples of solid convex sets were constructed, having Steiner polynomials with roots in the right half plane.

3. The explicit expressions for the Steiner and Weyl polynomials associated with the 'regular' convex sets.

Hereafter, we use the following identity for the Γ- function:

$$\Gamma(\zeta+1/2)\,\Gamma(\zeta+1)=\pi^{1/2}2^{-2\zeta}\Gamma(2\zeta+1)\,,\ \forall\zeta\in\mathbb{C}:2\zeta\neq-1,\,-2,\,-3,\,\dots\,. \tag{3.1}$$

Let as present explicit expressions for the Steiner polynomials associated with the 'regular' convex sets: balls, cubes, squeezed cylinders, as well as the expression for the Weyl polynomials associated with the boundary surfaces of these sets. The items related to balls are marked by the symbol $\odot$, the items related to cubes are marked by the symbol $\boxdot$.

$\odot$ The unit ball B^n.

Since $B^n+tB^n=(1+t)B^n$ for $t>0$, then, according to (1.8),

$$S^{\mathbb{R}^n}_{B^n}(t)=\mathrm{Vol}_n(B^n)\cdot(1+t)^n\,, \tag{3.2}$$

or

$$S^{\mathbb{R}^n}_{B^n}(t)=\mathrm{Vol}_n(B^n)\sum_{0\leq k\leq n}\frac{n!}{(n-k)!}\cdot\frac{t^k}{k!}\,. \tag{3.3}$$

Thus, the coefficients of the Steiner polynomial $S^{\mathbb{R}^n}_{B^n}$ for the ball B^n are:

$$s^{\mathbb{R}^n}_k(B^n)=\mathrm{Vol}_n(B^n)\cdot\frac{n!}{(n-k)!}\cdot\frac{1}{k!}\,,\quad 0\leq k\leq n\,. \tag{3.4}$$

$\odot$ The squeezed spherical cylinder $B^n\times 0$.

The Steiner polynomial for the squeezed spherical cylinder $B^n\times 0$ is:

$$S^{\mathbb{R}^{n+1}}_{B^n\times 0}(t)=\mathrm{Vol}_n(B^n)\cdot t\sum_{0\leq k\leq n}\frac{n!}{(n-k)!}\frac{\pi^{1/2}\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\frac{1}{k!}\,t^k\,. \tag{3.5}$$

The expression (3.5) is derived from (3.3) and (1.31)-(1.32). (See Lemma 1.15.)

Thus, the coefficients of the Steiner polynomial $S^{\mathbb{R}^{n+1}}_{B^n\times 0}$ for the squeezed spherical cylinder $B^n\times 0$ are:

$$s^{\mathbb{R}^{n+1}}_0(B^n\times 0)=0,\quad s^{\mathbb{R}^{n+1}}_{k+1}(B^n\times 0)=$$
$$=\mathrm{Vol}_n(B^n)\cdot\frac{n!}{(n-k)!}\cdot\frac{\pi^{1/2}\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\frac{1}{k!}\,,\quad 0\leq k\leq n\,. \tag{3.6}$$

$\odot$ The unit sphere $S^n=\partial B^{n+1}$.

According to (1.38) and (3.4), the Weyl coefficients of the n- dimensional sphere $S^n=\partial B^{n+1}$ are:

$$w_{2l}(\partial B^{n+1})=\mathrm{Vol}_n(\partial B^{n+1})\cdot\frac{n!}{(n-2l)!}\cdot\frac{1}{l!}\frac{1}{2^l}\,,\quad 0\leq l\leq[\tfrac{n}{2}]\,. \tag{3.7}$$

Thus, the Weyl polynomials associated with the n- dimensional sphere are:

$$W^{p}_{\partial B^{n+1}}(t)=\mathrm{Vol}_n(\partial B^{n+1})\cdot \\ \cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{1}{l!}\cdot\Big(\frac{t^2}{2}\Big)^l,\quad p=1,\,2,\,\dots\,. \tag{3.8}$$

$$W^{\infty}_{\partial B^{n+1}}(t)=\mathrm{Vol}_n(\partial B^{n+1})\cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{1}{l!}\cdot\Big(\frac{t^2}{2}\Big)^l\cdot \tag{3.9}$$

$\odot$ The boundary surface $\partial(B^n\times 0)$ of the squeezed spherical cylinder $B^n\times 0$.
According to (1.39) and (3.4), the Weyl coefficients of the n- dimensional improper surface $\partial(B^n\times 0)$ are:

$$w_{2l}(\partial(B^n\times 0))=\mathrm{Vol}_n(\partial(B^n\times 0))\cdot\frac{n!}{(n-2l)!}\cdot\frac{\Gamma(1/2)}{\Gamma(l+1/2)}\,\frac{1}{2^l},\quad 0\le l\le[\tfrac{n}{2}]\,. \tag{3.10}$$

Thus, the Weyl polynomials associated with the (improper) surface $\partial(B^n\times 0)$ are:

$$W^{p}_{\partial(B^n\times 0)}(t)=\mathrm{Vol}_n(\partial(B^n\times 0))\cdot \\ \cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{\Gamma(1/2)}{\Gamma(l+1/2)}\cdot\Big(\frac{t^2}{2}\Big)^l,\quad p=1,\,2,\,\dots\,. \tag{3.11}$$

$$W^{\infty}_{\partial(B^{n+1}\times 0)}(t)=\mathrm{Vol}_n(\partial(B^{n+1}\times 0))\cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{\Gamma(1/2)}{\Gamma(l+1/2)}\cdot\Big(\frac{t^2}{2}\Big)^l\cdot \tag{3.12}$$

$\boxdot$ The unit cube Q^n.
The Steiner polynomial $S^{\mathbb{R}^n}_{Q^n}$ is:

$$S^{\mathbb{R}^n}_{Q^n}(t)=\mathrm{Vol}_n(Q^n)\cdot\sum_{0\le k\le n}\frac{n!}{(n-k)!}\frac{1}{\Gamma(\frac{k}{2}+1)k!}\Big(\frac{\sqrt{\pi}}{2}\Big)^k t^k\,. \tag{3.13}$$

Expression (3.13) is obtained in the following way. The n- dimensional cube Q^n is considered as the Cartesian product of the one-dimensional cubes:

$$Q^n=Q^1\times\,\cdots\,\times Q^1\,.$$

For $n=1$, the Steiner polynomial is: $S^{\mathbb{R}^1}_{Q^1}(t)=2(1+t)$. We here use the fact that the Steiner polynomial of a Cartesian product can be expressed in terms of the Steiner polynomials belonging to the Cartesian factors. (See details in Section 12.)

We find, the coefficients of the Steiner polynomial for the cube Q^n to be given by:

$$s^{\mathbb{R}^n}_k(Q^n)=\mathrm{Vol}_n(Q^n)\cdot\frac{n!}{(n-k)!}\cdot\frac{1}{\Gamma(\frac{k}{2}+1)k!}\Big(\frac{\sqrt{\pi}}{2}\Big)^k,\quad 0\le k\le n\,. \tag{3.14}$$

⊡ The squeezed cubic cylinder $Q^n \times 0$.
The Steiner polynomial $S^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ is:

$$S^{\mathbb{R}^{n+1}}_{Q^n\times 0}(t) = \mathrm{Vol}_n(Q^n)\cdot t\sum_{0\le k\le n}\frac{n!}{(n-k)!}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2}+1)k!}\Big(\frac{\sqrt{\pi}}{2}\Big)^k t^k\,. \tag{3.15}$$

The expression (3.15) is derived from (3.13) and (1.31)-(1.32). (See Lemma 1.15.)

Thus, the coefficients of the Steiner polynomial $S^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ for the squeezed cubic cylinder are:

$$\begin{aligned} &s^{\mathbb{R}^{n+1}}_0(Q^n\times 0) = 0,\quad s^{\mathbb{R}^{n+1}}_{k+1}(Q^n\times 0) = \\ &\qquad = \mathrm{Vol}_n(Q^n)\cdot\frac{n!}{(n-k)!}\cdot\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2}+1)}\frac{1}{k!}\Big(\frac{\sqrt{\pi}}{2}\Big)^k,\quad 0\le k\le n\,. \end{aligned} \tag{3.16}$$

⊡ The boundary surface ∂Q^{n+1} of the unit cube Q^{n+1}.
According to (1.38) and (3.14), the Weyl coefficients of the n-dimensional surface ∂Q^{n+1} are:

$$\begin{aligned} &w_{2l}(\partial Q^{n+1}) = \mathrm{Vol}_n(\partial Q^{n+1})\cdot\frac{n!}{(n-2l)!}\cdot \\ &\qquad\cdot\frac{1}{\Gamma(l+\frac{1}{2}+1)}\frac{1}{2^l\,l!}\Big(\frac{\sqrt{\pi}}{2}\Big)^{2l+1},\quad 0\le l\le [\tfrac{n}{2}]\,. \end{aligned} \tag{3.17}$$

Taking into account the identity $\Gamma(l+1+\frac{1}{2})\cdot\Gamma(l+1) = \pi^{1/2}2^{-(2l+1)}\Gamma(2l+2)$, which is obtained by setting $\zeta = l+1/2$ in (3.1), we can rewrite the equality (3.17):

$$w_{2l}(\partial Q^{n+1}) = \mathrm{Vol}_n(\partial Q^{n+1})\cdot\frac{n!}{(n-2l)!}\cdot\frac{1}{(2l+1)!}\Big(\frac{\pi}{2}\Big)^l,\quad 0\le l\le [\tfrac{n}{2}]\,. \tag{3.18}$$

Thus, the Weyl polynomials associated with the n-dimensional surface ∂Q^{n+1} are:

$$\begin{aligned} &W^p_{\partial Q^{n+1}}(t) = \mathrm{Vol}_n(\partial Q^{n+1})\cdot \\ &\qquad\cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{1}{(2l+1)!}\cdot\Big(\frac{\pi t^2}{2}\Big)^l,\quad p=1,2,\ldots\,. \end{aligned} \tag{3.19}$$

$$W^\infty_{\partial Q^{n+1}}(t) = \mathrm{Vol}_n(\partial Q^{n+1})\cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{1}{(2l+1)!}\cdot\Big(\frac{\pi t^2}{2}\Big)^l\,. \tag{3.20}$$

⊡ The boundary surface $\partial(Q^n\times 0)$ of the squeezed cubic cylinder $Q^n\times 0$.
According to (1.39) and (3.14), the Weyl coefficients of the surface (improper) $\partial(Q^n\times 0)$ are:

$$\begin{aligned} &w_{2l}(\partial Q^n\times 0) = \mathrm{Vol}_n(\partial Q^n\times 0)\cdot\frac{n!}{(n-2l)!}\cdot \\ &\qquad\cdot\frac{\sqrt{\pi}}{\Gamma(l+\frac{1}{2})}\frac{1}{l!\,2^l}\Big(\frac{\pi}{2}\Big)^l,\quad 0\le l\le [\tfrac{n}{2}]\,. \end{aligned} \tag{3.21}$$

Using the identity $\Gamma(l+1/2)\Gamma(l+1) = \sqrt{\pi}2^{-2l}\Gamma(2l+1)$, which is obtained by setting $\zeta = l$ in (3.1), the equality (3.21) can be rewritten as follows:

$$w_{2l}(\partial Q^n \times 0) = \mathrm{Vol}_n(\partial Q^n \times 0) \cdot \frac{n!}{(n-2l)!} \cdot \frac{1}{(2l)!} \left(\frac{\pi}{2}\right)^{2l}, \quad 0 \leq l \leq [\tfrac{n}{2}]. \quad (3.22)$$

Thus, the Weyl polynomials associated with the improper n-dimensional surface $\partial(Q^n \times 0)$ are:

$$W^{p}_{\partial(Q^n\times 0)}(t) = \mathrm{Vol}_n(\partial(Q^n \times 0)) \cdot \\ \cdot \sum_{l=0}^{[\frac{n}{2}]} \frac{n!}{(n-2l)!} \cdot \frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)} \cdot \frac{1}{(2l)!} \cdot \left(\frac{\pi t^2}{2}\right)^{l}, \quad p = 1, 2, \ldots. \quad (3.23)$$

$$W^{\infty}_{\partial(Q^n\times 0)}(t) = \mathrm{Vol}_n(\partial(Q^n \times 0)) \cdot \sum_{l=0}^{[\frac{n}{2}]} \frac{n!}{(n-2l)!} \cdot \frac{1}{(2l)!} \cdot \left(\frac{\pi t^2}{2}\right)^{l}. \quad (3.24)$$

4. Weyl and Steiner polynomials of ‘regular’ convex sets as renormalized Jensen polynomials.

It would be difficult to directly investigate the location of the roots for the Steiner polynomials $S^{\mathbb{R}^n}_{B^n}$, $S^{\mathbb{R}^{n+1}}_{B^n\times 0}$, $S^{\mathbb{R}^n}_{Q^n}$, $S^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ and Weyl polynomials $W^{p}_{\partial B^{n+1}}$, $W^{p}_{\partial(B^n\times 0)}$, $W^{\infty}_{\partial Q^{n+1}}$, $W^{p}_{\partial(Q^n\times 0)}$ for a *finite* n. It turns out to be much easier first to investigate the roots of the entire functions, which are the limits of the (renormalized) Steiner and Weyl polynomials as $n \to \infty$. The properties of the roots belonging to the original Steiner and Weyl polynomials can then be deduced from the properties of these limiting entire functions.

Jensen polynomials. From the explicit expressions (3.3), (3.5), (3.13), (3.15) for the Steiner polynomials and (3.8), (3.9), (3.11), (3.12), (3.19), (3.20), (3.23), (3.24) for the Weyl polynomials we notice that each of these expressions contains the factor $\dfrac{n!}{(n-k)!}$, which is ‘a part’ of the binomial coefficient $\binom{n}{k}$. This factor can be expressed as

$$\frac{n!}{(n-k)!} = 1 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdot \ \cdots \ \cdot \left(1 - \frac{k-1}{n}\right) \cdot n^k, \quad 1 \leq k \leq n. \quad (4.1)$$

Definition 4.1.

1. Given a formal power series f:

$$f(t) = \sum_{0 \leq l < \infty} a_l t^l, \quad (4.2)$$

we define the following sequence of polynomials $\mathfrak{J}_n(f;t)$, $n = 1, 2, 3, \ldots$:

$$\mathfrak{J}_n(f;t) = \sum_{0\le l\le n} \frac{n!}{(n-l)!}\frac{1}{n^l}\cdot a_l t^l, \tag{4.3}$$

or, rewriting the factor $\frac{n!}{(n-l)!}\frac{1}{n^l}$,

$$\mathfrak{J}_n(f;t) = a_0 + \sum_{1\le l\le n} 1\big(1-\tfrac{1}{n}\big)\big(1-\tfrac{2}{n}\big)\,\cdots\,\big(1-\tfrac{l-1}{n}\big)\,\cdot a_l t^l. \tag{4.4}$$

The polynomials $\mathfrak{J}_n(f;t)$ are said to be the Jensen polynomials associated with the power series f.

2. Given a holomorphic *function* f in the disc $\{t: |t|<R\}$, where $R \le \infty$, we associate the sequence of the Jensen polynomials with the Taylor series (4.2) of the function f according to (4.3). We denote these polynomials by $\mathfrak{J}_n(f;t)$ as well and call them the Jensen polynomials associated with the function f.

3. The factors

$$j_{n,0} = 1, \quad j_{n,k} = 1\big(1-\tfrac{1}{n}\big), \big(1-\tfrac{2}{n}\big)\,\cdots\,\big(1-\tfrac{k-1}{n}\big),\ 1\le k\le n,$$
$$j_{n,k} = 0,\ k > n\,, \tag{4.5}$$

are said to be the Jensen multipliers.

Thus, the Jensen polynomials associated with an f of the form (4.2) can be written as:

$$\mathfrak{J}_n(f;t) = \sum_{0\le<\infty} j_{n,l}\cdot a_l t^l\,. \tag{4.6}$$

Since $j_{n,k}\to 1$ as k is fixed, $n\to\infty$, the following result is evident:

Lemma 4.2 (The approximation property of Jensen polynomials). *Given the power series* (4.2)*, the following statements hold:*

1. *The sequence of the Jensen polynomials $\mathfrak{J}_n(f;t)$ converges to the series f coefficients-wise;*
2. *If, moreover, the radius of convergence of the power series* (4.2) *is positive, say equal to R, $0 < R \le \infty$, then the sequence of the Jensen polynomials $\mathfrak{J}_n(f;t)$ converges to the function which is the sum of this power series locally uniformly in the disc $\{t: |t|<R\}$.*

The approximation property in not specific to the polynomials constructed from the *Jensen multipliers* $j_{n,k}$. This property holds for *any* multipliers $j_{n,k}$, which satisfy the conditions $j_{n,k}\to 1$ as k is fixed, $n\to\infty$, and are uniformly bounded: $\sup\limits_{k,n}|j_{n,k}|<\infty$. What is much more specific, that for some f, the polynomial $\mathfrak{J}_n(f;t)$ constructed from the *Jensen multipliers* $j_{n,k}$ preserve the property of f to possess only real roots.

Theorem (Jensen). *Let f be a* polynomial *such that all its roots are real. Then for each n, all roots of the Jensen polynomial $\mathfrak{J}_n(f,\,t)$ are real as well.*

This result is a special case of the Schur Composition Theorem [Schu1]. Actually, Jensen, [Jen], obtained a more general result in which formulation f can be, not only a polynomial with real roots, but also an entire function belonging to the *Laguerre-Pólya class*. We return to this generalization later, in Section 5. Now we focus our attention on the representation of the Steiner and Weyl polynomials as Jensen polynomials of certain entire functions.

The relation (4) as well as the expressions (3.3), (3.5), (3.13), (3.15) for the Steiner polynomials suggest us how the Steiner polynomials should be renormalized so that the renormalized polynomials tend to a non-trivial limit as $n \to \infty$.

Entire functions which generate the Steiner polynomials for balls, cubes, spherical and cubic cylinders. Let us introduce the infinite power series:

$$\mathcal{M}_{B^\infty}(t) = \sum_{0 \le k < \infty} \frac{1}{k!}\, t^k\,; \tag{4.7a}$$

$$\mathcal{M}_{B^\infty \times 0}(t) = \sum_{0 \le k < \infty} \frac{\Gamma(\frac{1}{2}+1)\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\, \frac{1}{k!}\, t^k\,; \tag{4.7b}$$

$$\mathcal{M}_{Q^\infty}(t) = \sum_{0 \le k < \infty} \frac{1}{\Gamma(\frac{k}{2}+1)k!} \Big(\frac{\sqrt{\pi}}{2}\Big)^k\, t^k\,; \tag{4.7c}$$

$$\mathcal{M}_{Q^\infty \times 0}(t) = \sum_{0 \le k < \infty} \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{k+1}{2}+1)k!} \Big(\frac{\sqrt{\pi}}{2}\Big)^k\, t^k\,. \tag{4.7d}$$

The series (4.7) represent entire functions which grow not faster than exponentially. More precisely, the functions $\mathcal{M}_{B^\infty}$ and $\mathcal{M}_{B^\infty \times 0}$ grow exponentially: they are of order 1 and of normal type, the functions $\mathcal{M}_{Q^\infty}$ and $\mathcal{M}_{BQ^\infty \times 0}$ grow subexponentially: they are of order 2/3 and of normal type.

For each of the entire functions (4.7) we consider the associated sequence of Jensen polynomials:

$$\mathcal{M}_{B^n}(t) = \mathfrak{J}_n(\mathcal{M}_{B^\infty};t)\,, \tag{4.8a}$$

$$\mathcal{M}_{B^n \times 0}(t) = \mathfrak{J}_n(\mathcal{M}_{B^\infty \times 0};t)\,, \tag{4.8b}$$

$$\mathcal{M}_{Q^n}(t) = \mathfrak{J}_n(\mathcal{M}_{Q^\infty};t) \tag{4.8c}$$

$$\mathcal{M}_{Q^n \times 0}(t) = \mathfrak{J}_n(\mathcal{M}_{Q^\infty \times 0};t)\,, \tag{4.8d}$$

From the expressions (3.3), (3.5), (3.13), (3.15) for the Steiner polynomials, it follows that they are related to the above introduced polynomials (4.8) as follows:

$$S^{\mathbb{R}^n}_{B^n}(t) = \mathrm{Vol}_n(B^n) \qquad \mathcal{M}_{B^n}(nt)\,; \tag{4.9a}$$
$$S^{\mathbb{R}^{n+1}}_{B^n\times 0}(t) = \mathrm{Vol}_n(B^n)\,\omega_1 t\,\mathcal{M}_{B^n\times 0}(nt)\,; \tag{4.9b}$$
$$S^{\mathbb{R}^n}_{Q^n}(t) = \mathrm{Vol}_n(Q^n) \qquad \mathcal{M}_{Q^n}(nt)\,; \tag{4.9c}$$
$$S^{\mathbb{R}^{n+1}}_{Q^n\times 0}(t) = \mathrm{Vol}_n(Q^n)\,\omega_1 t\,\mathcal{M}_{Q^n\times 0}(nt)\,; \tag{4.9d}$$

The polynomials $\mathcal{M}_{B^n}$, $\mathcal{M}_{B^n\times 0}$, $\mathcal{M}_{Q^n}$, $\mathcal{M}_{Q^n\times 0}$ can be interpreted as *renormalized Steiner polynomials*. We take the equalities (4.9) as the definition of the renormalized Steiner polynomials $\mathcal{M}_{B^n}$, $\mathcal{M}_{B^n\times 0}$, $\mathcal{M}_{Q^n}$, $\mathcal{M}_{Q^n\times 0}$ in terms of the 'original' Steiner polynomials $S^{\mathbb{R}^n}_{B^n}$, $S^{\mathbb{R}^{n+1}}_{B^n\times 0}$, $S^{\mathbb{R}^n}_{Q^n}$, and $S^{\mathbb{R}^{n+1}}_{Q^n\times 0}$.

From the approximation property of Jensen polynomials and from (4.8) it follows that

$$\mathcal{M}_{B^n}(t) \to \mathcal{M}_{B^\infty}(t), \quad \mathcal{M}_{B^n\times 0}(t) \to \mathcal{M}_{B^\infty\times 0}(t), \quad \mathcal{M}_{Q^n}(t) \to \mathcal{M}_{Q^\infty}(t),$$
$$\mathcal{M}_{Q^n\times 0}(t) \to \mathcal{M}_{Q^\infty\times 0}(t) \quad \text{as} \quad n \to \infty\,. \tag{4.10}$$

This explains the notation (4.7).

We summarize the above stated consideration as follows

Theorem 4.3. *Let $\{V^n\}$ be one of the four families of convex sets: $\{B^n\}$, $\{B^n\times 0\}$, $\{Q^n\}$, $\{Q^n\times 0\}$. For each of these four families, there exists an entire function*[16] *$\mathcal{M}_{V^\infty}$ such that in* any *dimension n, the renormalized Steiner polynomials $\mathcal{M}_{V^n}$, defined by (4.9), are generated by this entire function $\mathcal{M}_{V^\infty}$ to be the Jensen polynomials $\mathfrak{J}_n(\mathcal{M}_{V^\infty})$: for all n the equalities (4.8) hold.*

[16]The symbol V^∞ denotes whichever of the sets $\{B^\infty\}$, $\{B^\infty\times 0\}$, $\{Q^\infty\}$, $\{Q^\infty\times 0\}$ the case demands.

Entire functions which generate the Weyl polynomials for the surfaces of balls, cubes, spherical and cubic cylinders. Let us introduce the infinite power series:

$$\mathcal{W}^{p}_{\partial B^\infty}(t)=\sum_{l=0}^{\infty}\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{1}{l!}\cdot\Big(-\frac{t^2}{2}\Big)^{l},\quad p=1,\,2,\,\ldots\,; \tag{4.11a}$$

$$\mathcal{W}^{\infty}_{\partial B^\infty}(t)=\sum_{l=0}^{\infty}\frac{1}{l!}\cdot\Big(-\frac{t^2}{2}\Big)^{l}; \tag{4.11b}$$

$$\mathcal{W}^{p}_{\partial(B^\infty\times 0)}(t)=\sum_{l=0}^{\infty}\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{\Gamma(1/2)}{\Gamma(l+1/2)}\cdot\Big(-\frac{t^2}{2}\Big)^{l},\quad p=1,\,2,\,\ldots\,; \tag{4.11c}$$

$$\mathcal{W}^{\infty}_{\partial(B^\infty\times 0)}(t)=\sum_{l=0}^{\infty}\frac{\Gamma(1/2)}{\Gamma(l+1/2)}\cdot\Big(-\frac{t^2}{2}\Big)^{l}; \tag{4.11d}$$

$$\mathcal{W}^{p}_{\partial Q^\infty}(t)=\sum_{l=0}^{\infty}\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{1}{(2l+1)!}\cdot\Big(-\frac{\pi t^2}{2}\Big)^{l},\quad p=1,\,2,\,\ldots\,; \tag{4.11e}$$

$$\mathcal{W}^{\infty}_{\partial Q^\infty}(t)=\sum_{l=0}^{\infty}\frac{1}{(2l+1)!}\cdot\Big(-\frac{\pi t^2}{2}\Big)^{l}; \tag{4.11f}$$

$$\mathcal{W}^{p}_{\partial(Q^\infty\times 0)}(t)=\sum_{l=0}^{\infty}\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{1}{(2l)!}\cdot\Big(-\frac{\pi t^2}{2}\Big)^{l},\quad p=1,\,2,\,\ldots\,; \tag{4.11g}$$

$$\mathcal{W}^{\infty}_{\partial(Q^\infty\times 0)}(t)=\sum_{l=0}^{\infty}\frac{1}{(2l)!}\cdot\Big(-\frac{\pi t^2}{2}\Big)^{l}. \tag{4.11h}$$

The series (4.11) represent entire functions. The functions (4.11b) and (4.11d) are of order 2 and normal type, the functions (4.11a), (4.11c), (4.11f) and (4.11h) are of order 1 and normal type, the functions (4.11e) and (4.11g) are of order 2/3 and normal type.

For each of the entire functions (4.11) we consider the assotiated sequence of Jensen polynomials:

$$\mathcal{W}^{p}_{\partial B^{n+1}}(t)=\mathcal{J}_{2[n/2]}(\mathcal{W}^{p}_{\partial B^\infty};t)\,,\qquad 1\leq p\leq\infty\,; \tag{4.12a}$$

$$\mathcal{W}^{p}_{\partial(B^{n+1}\times 0)}(t)=\mathcal{J}_{2[n/2]}(\mathcal{W}^{p}_{\partial(B^\infty\times 0)};t)\,,\qquad 1\leq p\leq\infty\,; \tag{4.12b}$$

$$\mathcal{W}^{p}_{\partial Q^{n}}(t)=\mathcal{J}_{2[n/2]}(\mathcal{W}^{p}_{\partial Q^\infty};t)\,,\qquad 1\leq p\leq\infty\,; \tag{4.12c}$$

$$\mathcal{W}^{p}_{\partial(Q^{n}\times 0)}(t)=\mathcal{J}_{2[n/2]}(\mathcal{W}^{p}_{\partial(Q^\infty\times 0)};t)\,,\qquad 1\leq p\leq\infty\,. \tag{4.12d}$$

From the expressions (3.8), (3.9), (3.11), (3.12), (3.19), (3.20), (3.23), (3.24), for the Weyl polynomials it follows that they are related to the above introduced

polynomials (4.12) in the following way:

$$W^p_{\partial B^{n+1}}(t) = \quad \mathrm{Vol}_n(\partial B^{n+1}) \cdot \mathcal{W}^p_{\partial B^{n+1}}(int)\,; \tag{4.13a}$$
$$W^p_{\partial(B^n\times 0)}(t) = \mathrm{Vol}_n(\partial(B^n\times 0)) \cdot \mathcal{W}^p_{\partial(B^n\times 0)}(int)\,; \tag{4.13b}$$
$$W^p_{\partial Q^{n+1}}(t) = \quad \mathrm{Vol}_n(\partial Q^{n+1}) \cdot \mathcal{W}^p_{\partial Q^{n+1}}(int)\,; \tag{4.13c}$$
$$W^p_{\partial(Q^n\times 0)}(t) = \mathrm{Vol}_n(\partial(Q^n\times 0)) \cdot \mathcal{W}^p_{\partial(Q^n\times 0)}(int)\,; \tag{4.13d}$$

The equalities (4.13) hold for all $n:\ 1\leq n<\infty,\ p:\ 1\leq p\leq\infty$.

The polynomials $\mathcal{W}^p_{\partial B^{n+1}}$, $\mathcal{W}^p_{\partial(B^n\times 0)}$, $\mathcal{W}^p_{\partial Q^{n+1}}$, $\mathcal{W}^p_{\partial(Q^n\times 0)}$ can be interpreted as *renormalized Weyl polynomials*. We take the equalities (4.13) as the definition of the renormalized Weyl polynomials $\mathcal{W}^p_{\partial B^{n+1}}$, $\mathcal{W}^p_{\partial(B^n\times 0)}$, $\mathcal{W}^p_{\partial Q^{n+1}}$, $\mathcal{W}^p_{\partial(Q^n\times 0)}$ in terms of the 'original' Steiner polynomials $W^p_{\partial B^{n+1}}$, $W^p_{\partial(B^n\times 0)}$, $W^p_{\partial Q^{n+1}}$, $W^p_{\partial(Q^n\times 0)}$.

From the approximation property of Jensen polynomials and from (4.12) it follows that for every fixed $p,\ 1\leq p\leq\infty$,

$$\begin{gathered}\mathcal{W}^p_{\partial B^{n+1}}(t)\to\mathcal{W}^p_{\partial B^\infty}(t),\quad \mathcal{W}^p_{\partial(B^n\times 0)}(t)\to\mathcal{W}^p_{\partial(B^\infty\times 0)}(t),\\ \mathcal{W}^p_{\partial Q^{n+1}}(t)\to\mathcal{W}^p_{\partial Q^\infty}(t), \mathcal{W}^p_{\partial(Q^n\times 0)}(t)\to\mathcal{W}^p_{\partial(Q^\infty\times 0)}(t)\ \text{ as }\ n\to\infty\,.\end{gathered} \tag{4.14}$$

This explains the notation (4.11).

We summarize the above as follows.

Theorem 4.4. *Let $\{\mathcal{M}^n\}$ be one of the four families of n-dimensional convex surfaces: $\{\partial B^{n+1}\}$, $\{\partial(B^n\times 0)\}$, $\{\partial Q^{n+1}\}$, $\{\partial(Q^n\times 0)\}$. For each of these four families, and for each p, $1\leq p\leq\infty$, there exists an entire function*[17] $\mathcal{W}^p_{\mathcal{M}^\infty}$ *such that in any dimension n, the renormalized Weyl polynomials $\mathcal{W}^p_{\mathcal{M}^n}$, defined by* (4.13)*, are generated by this entire function $\mathcal{W}^p_{\mathcal{M}^\infty}$ to be the Jensen polynomials $\mathcal{J}_2[n/2](\mathcal{W}^p_{\mathcal{M}^\infty})$.*

5. Entire functions of the Hurwitz and of the Laguerre-Pólya class. Multipliers preserving location of roots.

Hurwitz class of entire functions.

Definition 5.1. An entire function H is said to be in the Hurwitz class, written $H\in\mathcal{H}$, if

1. $H\not\equiv 0$, and the real part of any root of H is strictly negative, i.e. if $H(\zeta)=0$, then $\mathrm{Re}\,\zeta<0$.
2. The function H is of exponential type: $\overline{\lim\limits_{|z|\to\infty}}\frac{\ln|H(z)|}{|z|}<\infty$, and its defect d_H is non-negative: $d_H\geq 0$, where
$$2d_H=\overline{\lim_{r\to+\infty}}\frac{\ln|H(r)|}{r}-\overline{\lim_{r\to+\infty}}\frac{\ln|H(-r)|}{r}\,. \tag{5.1}$$

[17] The symbol $\mathcal{M}^\infty$ denotes whichever of the sets $\{\partial B^\infty\}$, $\{\partial(B^\infty\times 0)\}$, $\{\partial Q^\infty\}$, $\{\partial(Q^\infty\times 0)\}$ the case demands.

The following functions are examples of entire functions in class $\mathcal{H}$:
a). A dissipative polynomial $P(t)$.
b). An exponential $\exp\{\alpha t\}$, where $\operatorname{Re}\alpha \geq 0$.
c). The product $P(t)\cdot\exp\{\alpha t\}$: $P(t)$ is a dissipative polynomial, $\operatorname{Re}\alpha \geq 0$.

The significance of the Hurwitz class of entire functions stems from the fact that functions in this class [18] are locally uniform limits in $\mathbb{C}$ of dissipative polynomials.

Laguerre-Pólya class of entire functions.

Definition 5.2. An entire function E is said to be in the Laguerre-Pólya class, written $E \in \mathcal{L}\text{-}\mathcal{P}$, if E is real and can be expressed in the form

$$E(t) = ct^n e^{-\beta t^2+\alpha t} \prod_{k=1}^{\infty} (1+t\alpha_k)\, e^{-t\alpha_k}, \tag{5.2}$$

where $c \in \mathbb{R}\setminus 0$, $\beta \geq 0$, $\alpha \in \mathbb{R}$, $\alpha_k \in \mathbb{R}$, n is non-negative integer, and $\sum_{k=1}^{\infty}\alpha_k^2 < \infty$.

A function $E \in \mathcal{L}\text{-}\mathcal{P}$ is said to be of type I, denoted $E \in \mathcal{L}\text{-}\mathcal{P}\text{-}\mathrm{I}$, if it is expressible in the form:

$$E(t) = ct^n e^{\alpha t} \prod_{k=1}^{\infty} (1+t\alpha_k), \tag{5.3}$$

where $c \in \mathbb{R}\setminus 0$, $\alpha \geq 0$, $\alpha_k \geq 0$, n is non-negative integer, and $\sum_{k=1}^{\infty}\alpha_k < \infty$.

The significance of the Laguerre-Pólya class stems from the fact that functions in this class, *and only these*, are the locally uniform limits in $\mathbb{C}$ of polynomials with only real roots. (See [Lev1, Chapter 8], [Obr, Chapter II, Theorems 9.1, 9.2, 9.3].)

Lemma 5.3. *An entire function E, which is in the Laguerre-Pólya class of type* I *also belongs to the Hurwitz class:*

$$\mathcal{L}\text{-}\mathcal{P}\text{-}\mathrm{I} \subset \mathcal{H}.$$

Proof. The roots of the entire function E which admit the representation (5.3) are located at the points $-(\alpha_k)^{-1}$, thus they are strictly negative. From the properties of the infinite product $\prod\limits_{k=1}^{\infty}(1+t\alpha_k)$ with $\sum_{k=1}^{\infty}|\alpha_k| < \infty$, it follows that a function E which admits the representation (5.3) is of exponential type α, and $\overline{\lim\limits_{r\to+\infty}} \frac{\ln|H(\pm r)|}{r} = \pm\alpha$. Thus, the defect $d_H = \alpha \geq 0$ since $\alpha \geq 0$. □

[18] The full description of the class of entire functions which are the limits of dissipative polynomials can be found in [Lev1, Chapter VIII, Theorem 4]. This class (up to the change of variables $z \to iz$) is denoted by P^* there.

Multipliers preserving the reality of roots.

Definition 5.4. A sequence $\{\gamma_k\}_{0\leq k<\infty}$ of real numbers is a P-S multiplier sequence[19] if for every polynomial f:

$$f(t)=\sum_{0\leq k\leq n} a_k t^k$$

with only real roots, the polynomial

$$h(t)=\sum_{0\leq k\leq n} \gamma_k a_k t^k$$

has only real roots as well. (The degree n of the polynomial f can be arbitrary.)

Theorem (Pólya, Schur). *A sequence $\{\gamma_k\}_{0\leq k<\infty}$ of real numbers, $\gamma_k\not\equiv 0$, is a P-S multiplier sequence if and only if the power series*

$$\Psi(t)=\sum_{0\leq k\leq \infty} \frac{\gamma_k}{k!}\,t^k$$

represents an entire function, and either the function $\Psi(t)$ or the function $\Psi(-t)$ is in the Lagierre-Pólya class of type I.

This result was obtained in [PoSch].

Theorem 5.5. [Jensen-Craven-Csordas-Williamson.] *Let $E(t)$ be an entire function belonging to the Laguerre-Pólya class $\mathcal{L}$-$\mathcal{P}$, and $\{\mathcal{J}_n(E,t)\}_{n=1,2,3,\dots}$ be the sequence of the Jensen polynomials associated with the function E.* (Definition 4.1.)

1. *Then, for each n, all roots of the polynomial $\mathcal{J}_n(E,t)$ are real.*

2. *If $E(t)$ belongs to the subclass $\mathcal{L}$-$\mathcal{P}$-I of the Laguerre-Pólya class $\mathcal{L}$-$\mathcal{P}$, then for each n, all roots of the polynomial $\mathcal{J}_n(E,t)$ are negative.*

3. *If, moreover, $E(t)$ is not of the form $E(t)=p(t)\,e^{\beta t}$, where $p(t)$ is a polynomial, then for each n, all roots of the polynomial $\mathcal{J}_n(E,t)$ are simple .*

Statement 1 of the theorem was proved by Jensen[20], [Jen]. It is a special case of Theorem by G.Pólya and I. Schur obtaining by choosing $\Psi(t)=\Big(1+\dfrac{t}{n}\Big)^n$. Statement 3 is a refinement of Statement 1 due to G.Csordas and J. Williamson in [CsWi], who also offer an alternative proof of Statement 1. In [CsWi], the main Theorem is formulated on p. 263. It appears as the Statement 3 of Theorem 5.5 of the present paper. In [CsWi], this theorem was inaccurately formulated. A corrected version can be found in [CrCs3, Section 4.1].

[19] P-S stands for Pólya-Schur

[20] Though Jensen himself did not introduce explicitly the polynomials which are called 'the Jensen polynomials' now.

Theorem 5.6. *Let H be an entire function belonging to the Hurwitz class $\mathcal{H}$, and $\{\mathcal{J}_n(H,\,t)\}_{n=1,\,2,\,3,\,\dots}$ be the sequence of the Jensen polynomials associated with the function H.* (Definition 4.1.) *Then, for each n, the polynomial $\mathcal{J}_n(H,\,t)$ is dissipative.*

Theorem 5.6 can be obtained as a consequence of Theorem 5.5 and the Hermite-Biehler Theorem. A proof of Theorem 5.6 will be presented in Section 5.

Laguerre multipliers.

Theorem (Laguerre). *Let an entire function $E(t)$,*

$$E(t) = \sum_{0 \leq l < \omega} \varepsilon_l t^l, \quad \omega \leq \infty, \tag{5.4}$$

be in the Laguerre-Pólya class: $E \in \mathcal{L}\text{-}\mathcal{P}$. Furthermore, let ψ be an entire function be in the Laguerre-Pólya class $\mathcal{L}\text{-}\mathcal{P}$, such that all of its roots are negative.

1. *Then the power series*

$$E_\psi = \sum_{0 \leq l < \omega} \varepsilon_l \psi(l) t^l \tag{5.5}$$

   converges for every t and its sum is an entire function of the Laguerre-Pólya class: $E_\psi \in \mathcal{L}\text{-}\mathcal{P}$.
2. *If moreover $E(t)$ is of type* I*: $E \in \mathcal{L}\text{-}\mathcal{P}\text{-}\mathrm{I}$, then the the sum of power series* (5.5) *is also an entire function of type* I*: $E_\psi \in \mathcal{L}\text{-}\mathcal{P}\text{-}\mathrm{I}$.*

This theorem appeared in [Lag1, section 18, p.117], or [Lag, p. 202]. Laguerre himself formulated this theorem for a polynomial with real roots. The extended formulation, where E is a general entire function from the class $\mathcal{L}\text{-}\mathcal{P}$, can be found in the paper [PoSch, p. 112], or in its reprint in [Po, p.123]. In [PoSch] the extended formulation is attributed to Jensen, see [Jen].

The above mentioned results of Pólya, Schur, Laguerre, and Jensen, as well as of many related results, can be found in [Obr, Chapter II], [Lev1, Chapter VIII], [RaSc, Chapter 5, especially Sections 5.5, 5.6, 5.7], and in numerous papers of Th. Craven and G. Csordas (See for example [CrCs1]). See also [PoSz, Part five]. The book by L. de Branges [deBr] is also closely related to this group of problems.

6. Properties of entire functions generating Steiner and Weyl polynomials of 'regular' convex sets and their surfaces.

Entire functions generating the Steiner polynomials.

Theorem 6.1. *The entire functions* (4.7) *generating the renormalised Steiner polynomials of balls, cubes, squeezed spherical and cubic cylinders, possesses the following properties:*

1. *The function $\mathcal{M}_{B^\infty}$ is of type* I *of the Laguerre-Pólya class.*

2. *The function* $\mathcal{M}_{B^\infty\times 0}$ *belongs to the Hurwitz class. It has infinitely roots, and all but finitely many of its roots are non-real;*
3. *The function* $\mathcal{M}_{Q^\infty}$ *is of type* I *of the Laguerre-Pólya class;*
4. *The function* $\mathcal{M}_{Q^\infty\times 0}$ *is of type* I *of the Laguerre-Pólya class.*

Lemma 6.2. *The function* $\dfrac{1}{\Gamma(t+1)}$, *where* Γ *is the Euler Gamma function, is in the Laguerre-Pólya class and all its roots are negative.*

Indeed,

$$\frac{1}{\Gamma(t+1)} = e^{Ct}\prod_{1\le k<\infty}\left(1+\frac{t}{k}\right)e^{-\frac{t}{k}},$$

(C is the Euler constant, $C \approx 0.5772156\ldots$.)

Proof of Theorem 6.1. Statement 1 is evident: $\mathcal{M}_{B^\infty}(t) = e^t$.

To obtain Statement 3, we remark that the function $\mathcal{M}_{Q^\infty}$ is of the form E_ψ, (5.5), where $E(t) = \exp\{\frac{\sqrt{\pi}}{2}t\}$, and $\psi(t) = \dfrac{1}{\Gamma(\frac{t}{2}+1)}$. Then we apply the Laguerre theorem on multipliers to these E and ψ. The needed property of ψ is formulated as Lemma 6.2.

Statement 4 can be obtained in the same way as the statement 3. One need only take $E(t) = \exp\{\frac{\sqrt{\pi}}{2}t\}$, and $\psi(t) = \dfrac{\Gamma(\frac{1}{2})}{\Gamma(\frac{t+1}{2}+1)}$.

Proof of Statement 2 is more complicated. From (4.7b) it follows that

$$\mathcal{M}_{B^\infty\times 0}(t) = \sum_{0\le k<\infty} B(\tfrac{k}{2}+1,\tfrac{1}{2})\frac{1}{k!}\,t^k = \sum_{0\le k<\infty}\int_0^1 \xi^{\frac{k}{2}}(1-\xi)^{-\frac{1}{2}}\,d\xi\,\frac{1}{k!}\,t^k\,.$$

Changing the order of summation and integration and summing the exponential series, we obtain the integral representation:

$$\mathcal{M}_{B^\infty\times 0}(t) = 2\int_0^1 (1-\xi^2)^{-\frac{1}{2}}\xi e^{\xi t}d\xi\,. \tag{6.1}$$

The fact that the function $\mathcal{M}_{B^\infty\times 0}$ belongs to the Hurwitz class will be derived from the integral representation (6.1). This will be done in Section 13. □

Entire functions generating the Weyl polynomials.

Lemma 6.3. *Let*

$$E(t) = \sum_{0\le l<\infty} a_l t^{2l} \tag{6.2}$$

be an even entire function of the class $\mathcal{L}$-$\mathcal{P}$, *and let* $p>0$ *be a number. Then the function* $E_p(t)$ *defined by the power series*

$$E_p(t) \overset{\text{def}}{=} \sum_{1\le l<\infty}\frac{2^{-l}\Gamma\big(\frac{p}{2}+1\big)}{\Gamma\big(l+\frac{p}{2}+1\big)}\cdot a_l t^{2l}, \tag{6.3}$$

belongs to the class $\mathcal{L}$-$\mathcal{P}$ *as well.*

Proof. Lemma 6.3 is the consequence of the Laguerre theorem on multipliers. The function

$$\psi_p(t) = \frac{2^{-\frac{t}{2}}\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{t}{2}+\frac{p}{2}+1\right)} \tag{6.4}$$

is in the Laguerre-Pólya class (see Lemma 6.2), and its roots are negative. □

We point out, see (4.11), that the entire functions $\mathcal{W}^p_{\partial B^\infty}$, $\mathcal{W}^p_{\partial(B^\infty\times 0)}$, $\mathcal{W}^p_{\partial Q^\infty}$, $\mathcal{W}^p_{\partial(Q^\infty\times 0)}$, which generate the Weyl polynomials with finite index p for the appropriate families of convex surfaces, can be obtained from the entire functions $\mathcal{W}^\infty_{\partial B^\infty}$, $\mathcal{W}^\infty_{\partial(B^\infty\times 0)}$, $\mathcal{W}^\infty_{\partial Q^\infty}$, $\mathcal{W}^\infty_{\partial(Q^\infty\times 0)}$, which generate the Weyl polynomials with infinite index, by means of a transformation of the form

$$\sum_{0\le k<\infty} a_k t^k \to \sum_{0\le k<\infty} \psi_p(k)\, a_k t^k \,.$$

Theorem 6.4.

1. *The functions* $\mathcal{W}^\infty_{\partial B^\infty}$, $\mathcal{W}^\infty_{\partial Q^\infty}$, $\mathcal{W}^\infty_{\partial(Q^\infty\times 0)}$ *belong to the Laguerre-Pólya class* $\mathcal{L}$-$\mathcal{P}$.
2. *The function* $\mathcal{W}^\infty_{\partial(B^\infty\times 0)}$ *does not belong to the Laguerre-Pólya class* $\mathcal{L}$-$\mathcal{P}$*: this function has infinitely many non-real roots.*

Proof. Statement 1 is evident in view of the explicit expressions:

$$\mathcal{W}^\infty_{\partial B^\infty}(t) = \exp\{-t^2/2\}\,, \tag{6.5}$$

$$\mathcal{W}^\infty_{\partial Q^\infty}(t) = \frac{\sin\{(\pi/2)^{\frac{1}{2}}t\}}{(\pi/2)^{\frac{1}{2}}t}\,, \tag{6.6}$$

$$\mathcal{W}^\infty_{\partial(Q^\infty\times 0)} = \cos\{(\pi/2)^{\frac{1}{2}}\,t\}\,. \tag{6.7}$$

The function $\mathcal{W}^\infty_{\partial(B^\infty\times 0)}$, which appears in Statement 2, can not be expressed in terms of 'elementary' functions, but it can be expressed in terms of the Mittag-Leffler function $\mathcal{E}_{1,\frac{1}{2}}$:

$$\mathcal{W}^\infty_{\partial(B^\infty\times 0)}(t) = \sqrt{\pi}\,\mathcal{E}_{1,\frac{1}{2}}\left(-\frac{t^2}{2}\right), \tag{6.8}$$

where

$$\mathcal{E}_{\alpha,\beta}(z) = \sum_{0\le k<\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}\,. \tag{6.9}$$

The integral representation

$$\sqrt{\pi}\,\mathcal{E}_{1,\frac{1}{2}}(t) = 1 + t\int\limits_0^1 (1-\xi)^{-\frac{1}{2}} e^{t\xi}\, d\xi\,. \tag{6.10}$$

can be derived from (6.9). The integral representation (6.10) can be derived from the Taylor series (6.9) in the same way as the integral representation (6.1) was derived from the Taylor series (4.7b). From (6.10) the following asymptotic relations can be obtained:

$$\sqrt{\pi}\mathcal{E}_{1,\,\frac{1}{2}}(t) = \begin{cases} \frac{1}{2t}\,(1+o(1)), & t \to -\infty, \\ \sqrt{\pi t}\, e^t\,(1+o(1)), & t \to +\infty. \\ O(|t|), & t \to \pm i\infty. \end{cases} \tag{6.11}$$

From (6.11) it follows that the indicator diagram of the entire function $\mathcal{E}_{1,\,\frac{1}{2}}(t)$ of the exponential type is the interval $[0,\,1]$. Moreover, the function $\mathcal{E}_{1,\,\frac{1}{2}}(it)$ belongs to the class C, as this class was defined in [Lev2, Lecture 17]. It follows from the Cartwright-Levinson Theorem, which appears as Theorem 1 of the Lecture 17 in [Lev2]), that the function $\mathcal{E}_{1,\,\frac{1}{2}}(t)$ has infinitely many roots. These roots have a positive density, and are located 'near' the rays $\arg t = \frac{\pi}{2}$ and $\arg t = -\frac{\pi}{2}$. From this and from (6.8) it follows that the roots of the function $\mathcal{W}^{\infty}_{\partial(B^\infty\times 0)}(t)$ are located near four rays $\arg t = \frac{\pi}{4}$, $\arg t = \frac{3\pi}{4}$, $\arg t = \frac{5\pi}{4}$, $\arg t = \frac{7\pi}{4}$. In particular, infinitely many of the roots of the function $\mathcal{W}^{\infty}_{\partial(B^\infty\times 0)}(t)$ are non-real. □

Remark 6.5. Much more precise results about the Mittag-Leffler function $\mathcal{E}_{\alpha,\beta}$ and the distribution of its roots are available. See, for example, [EMOT, section 18.1], or [Djr].

Theorem 6.6.

1. *For every $p = 1, 2, \ldots,$ the functions $\mathcal{W}^p_{\partial B^\infty}$, $\mathcal{W}^p_{\partial Q^\infty}$, $\mathcal{W}^p_{\partial(Q^\infty\times 0)}$ belong to the Laguerre-Pólya class $\mathcal{L}$-$\mathcal{P}$.*
2. *If p is large enough, then the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ does not belong to the Laguerre-Pólya class $\mathcal{L}$-$\mathcal{P}$: it has non-real roots.*

Proof. Statement 1 of this theorem is a consequence of the statement 1 of Theorem 6.4 and Lemma 6.3. The statement 2 of this theorem is a consequence of statement 2 from Theorem 6.4 and the approximation property (1.43). □

Remark 6.7. The fact that the function $\mathcal{W}^p_{\partial B^\infty}$ belongs to the Laguerre-Pólya class $\mathcal{L}$-$\mathcal{P}$, i.e., the fact that all its roots are real, can be established without reference to Lemma 6.3. The function $\mathcal{W}^p_{\partial B^\infty}$ can be expressed in terms of Bessel functions J_ν. Recall that for arbitrary ν,

$$J_\nu(t) = \left(\frac{t}{2}\right)^{\nu} \sum_{0\le l<\infty} \frac{(-1)^l (t^2/4)^l}{l!\,\Gamma(\nu+l+1)}\,. \tag{6.12}$$

Comparing (6.12) with (4.11a), we see that

$$\mathcal{W}^p_{\partial B^\infty}(t) = \Gamma\left(\frac{p}{2}+1\right)\left(\frac{t}{2}\right)^{-\frac{p}{2}} J_{\frac{p}{2}}(t)\,. \tag{6.13}$$

In particular,
for [21] $p=1$,

$$\mathcal{W}^1_{\partial B^\infty}(t)=\frac{\sin t}{t}, \tag{6.14}$$

for $p=2$,

$$\mathcal{W}^2_{\partial B^\infty}(t)=2\,\frac{J_1(t)}{t}. \tag{6.15}$$

It is known that for every $\nu>-1$, all roots of the Bessel function $J_\nu(t)$ are real (This result is due to A.Hurwitz. See, for example, [Wat, Chapter XV, Section 15.27].)

The statement 2 of Theorem 6.6 can be further refined.

Theorem 6.8.

1. *For $p=1,\,2,\,4,$ the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ belongs to the Laguerre-Pólya class $\mathcal{L}$-$\mathcal{P}$;*
2. *For $p:\,5\le p\le\infty$, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ does not belong to the Laguerre-Pólya class $\mathcal{L}$-$\mathcal{P}$: it has infinitely many non-real roots.*

Proof. For every $p\ge 1$, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ admits the integral representation

$$\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)=p\int\limits_0^1(1-\xi^2)^{\frac{p}{2}-1}\xi\cos t\xi\,d\xi\,. \tag{6.16}$$

This integral representation can be obtained from (4.11c) in the same way that the integral representation (6.1) was obtained from (4.7b). Using the identity

$$\Gamma(l+1/2)\,\Gamma(l+1)=\Gamma(1/2)\,2^{-2l}\,\Gamma(2l+1),$$

in equation (4.11c), we obtain:

$$\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)=\frac{p}{2}\sum_{0\le l<\infty}\mathrm{B}(l+1,p/2)(-1)^l\frac{t^{2l}}{(2l)!}\,.$$

We use then the integral representation for the beta-function, change the order of summation and integration and sum the series using the Taylor expansion for $\cos z$. For every $p:\,1\le p<\infty$, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ can be calculated asymptotically. This calculation can be done using the integral representation (6.16). The asymptotic expression for the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ is presented in Section 13, see (13.27), (13.28). From this expression it follows that:
1. For $p>4$, infinitely many (more specifically, all but finitely many) roots of the $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ are non-real. This is sufficient for the negative result of the statement 2 of Theorem 6.8 to be obtained.

[21] Deriving (6.14) from (6.13), we used the formula $J_{\frac{1}{2}}(t)=\left(\frac{2}{\pi t}\right)^{\frac{1}{2}}\sin t$. (Concerning this formula, see, for example, [WhWa, section 17.24].) However, (6.14) may be obtained directly from (4.11a).

2. For $p \le 4$, all but finitely many roots of the function $\mathcal{W}^p_{\partial(B^\infty \times 0)}$ are real and simple. This alone is not sufficient to show Statement 1. For $p = 2$ and $p = 4$, the function $\mathcal{W}^p_{\partial(B^\infty \times 0)}$ can be calculated explicitly. The case $p = 3$ remains open. A proof of the fact that for $p = 1, 2, 4$ all roots of the function $\mathcal{W}^p_{\partial(B^\infty \times 0)}$ are real will be presented in Section 13. See Lemma 13.7. □

Proof of Theorem 2.9. According to Theorems 6.4, 6.6 and 6.8 (we here refer to the first statement of each of these theorems), each of the functions $\mathcal{W}^p_{\partial B^\infty}$, $\mathcal{W}^p_{\partial Q^\infty}$, $\mathcal{W}^p_{\partial(Q^\infty \times 0)}$ with $p: 1 \le p \le \infty$, and $\mathcal{W}^p_{\partial(B^\infty \times 0)}$ with $p = 1, 2, 4$, belongs to the Laguerre-Pólya class $\mathcal{L}$-$\mathcal{P}$. By Theorem 5.5, the Jensen polynomials associated with each of these entire functions, have only simple real roots. According to Theorem 4.4, the renormalized Weyl polynomials $\mathcal{W}^p_{\partial B^{n+1}}$, $\mathcal{W}^p_{\partial Q^{n+1}}$, $\mathcal{W}^p_{\partial(Q^n \times 0)}$ with $p: 1 \le p \le \infty$, and $\mathcal{W}^p_{\partial(B^n \times 0)}$ with $p = 1, 2, 4$ have only simple real roots. Because of the renormalizing relations (4.13), the Weyl polynomials $W^p_{\partial B^{n+1}}$, $W^p_{\partial Q^{n+1}}$, $W^p_{\partial(Q^n \times 0)}$ are conservative. □

Proof of Theorem 2.11. According to Theorem 6.8, Statement 2, for $p: 5 \le p \le \infty$, each of the entire functions $\mathcal{W}^p_{\partial(B^\infty \times 0)}$ has infinitely many non-real roots. Since for fixed p, $\mathcal{J}_n(\mathcal{W}^p_{\partial(B^\infty \times 0)}; t) \to \mathcal{W}^p_{\partial(B^\infty \times 0)}(t)$ locally uniformly in $\mathbb{C}$ as $n \to \infty$, the Hurwitz Theorem yields that every polynomial $\mathcal{J}_n(\mathcal{W}^p_{\partial(B^\infty \times 0)})$ with $p, n: p \ge 5$, $n \ge N(p)$ has non-real roots. By Theorem 4.4, $\mathcal{J}_n(\mathcal{W}^p_{\partial(B^\infty \times 0)}) = \mathcal{W}^p_{\partial(B^n \times 0)}$. Thus, the polynomial $\mathcal{W}^p_{\partial(B^n \times 0)}$ has non-real roots. Because of (4.12), the polynomial $W^p_{\partial(B^n \times 0)}$ has roots which do not belong to the imaginary axis. □

Proof of Lemma 2.4. This lemma is a consequence of lemma 6.3. If the polynomial $W^\infty_{\mathcal{M}}(t)$ is conservative, then the polynomial $E(t) = W^\infty_{\mathcal{M}}(it)$ is a real polynomial with only real simple roots. The function $E_p(t) = W^p_{\mathcal{M}}(it)$ is related to $E(t) = W^\infty_{\mathcal{M}}(it)$ as well as the function $E_p(t)$ from (6.3) is related to $E(t)$ from (6.2). By Lemma 6.3, all roots of E_p are real. Let us show that the roots are simple. Consider the function $E(t) + \varepsilon$, were ε is a small real number, positive or negative. Since all roots of the polynomial $E(t)$ are real and simple, all roots of the polynomial $E(t) + \varepsilon$ are real if $|\varepsilon|$ is small enough. By Lemma 6.3, all roots of the polynomial $E_p(t) + \varepsilon$ are real. However if the polynomial $E_p(t)$ has a multiple root, by applying the perturbation $E_p(t) \to E_p(t) + \varepsilon$ with an appropriate choice of sign for ε, this root splits into simple roots and some of these roots will be non-real. □

Remark 6.9. We apply Lemma 2.4 in the special cases $n = 2, 3, 4, 5$ only. In these cases the Lemma is quite elementary. Only the cases $n = 4$ and $n = 5$ deserve some attention. The cases $n = 2$ and $n = 3$ are trivial. The cases $n = 4$ and $n = 5$ are reduced to the following elementary statement:

Let w_0, w_2, w_4 be positive numbers. Assume that the roots of the polynomial $Q(t) = w_0 + w_2 t + w_4 t^2$ are negative and different. Then for every $p > 0$, the roots of the polynomial

$$Q^p(t) = w_0 + \frac{w_2}{(p+2)} t + \frac{w_4}{(p+2)(p+4)} t^2$$

are also negative and different.

Indeed, the conditions posed on polynomials Q and Q^p are equivalent to the inequalities

$$w_2^2 > w_0 w_4 \quad \text{and} \quad \left(\frac{w_2}{p+2}\right)^2 > w_0 \frac{w_2}{(p+2)(p+4)} \,.$$

It is evident that the first of these inequalities implies the second.

7. The Hermite-Biehler Theorem and its application.

In its traditional form, Hermite-Biehler Theorem gives conditions under which all roots of a polynomial belong to the upper half plane $\{z : \operatorname{Im} z > 0\}$. We need the version of this theorem adopted to the left half plane, and for the case of polynomials with non-negative coefficients only. Before we present out reformulation of the Hermite-Biehler Theorem, we give several definitions:

Definition 7.1. Let S_1 and S_2 be two sets which are situated on the same straight line[22] L of the complex plane: $S_1 \subset L,\ S_2 \subset L$, such that each of the sets S_1, S_2 consists only of isolated points. The sets S_1 and S_2 interlace if between every two points of S_1 there is a point of S_2, and between every two points of S_2 there is a point of S_1.

Definition 7.2. Let P be a power series:

$$P(t) = \sum_{0 \le k} p_k t^k, \tag{7.1}$$

where t is a complex variable and the coefficients p_k are complex numbers.

The real part ${}^{\mathcal{R}}P$ and the imaginary part ${}^{\mathcal{I}}P$ of P are defined as

$${}^{\mathcal{R}}P(t) = \frac{P(t) + \overline{P(\overline{t})}}{2}, \quad {}^{\mathcal{I}}P(t) = \frac{P(t) - \overline{P(\overline{t})}}{2i}\,, \tag{7.2}$$

where the overline bar is used as a notation for complex conjugation.

The even part ${}^{\mathcal{E}}P$ and the odd part ${}^{\mathcal{O}}P$ of P are defined as

$${}^{\mathcal{E}}P(t) = \frac{P(t) + P(-t)}{2}, \quad {}^{\mathcal{O}}P(t) = \frac{P(t) - P(-t)}{2}\,, \tag{7.3}$$

In term of coefficients,

$${}^{\mathcal{R}}P(t) = \sum_{0 \le k} a_k t^k, \quad {}^{\mathcal{I}}P(t) = \sum_{0 \le k} b_k t^k, \tag{7.4a}$$

[22] In our considerations the straight line L will be either the real axis or the imaginary axis.

where

$$a_k = \frac{p_k + \overline{p_k}}{2}, \quad b_k = \frac{p_k - \overline{p_k}}{2i}\,, \tag{7.4b}$$

and

$$^{\mathcal{E}}P(t) = \sum_{0 \leq l} p_{2l} t^{2l}, \quad ^{\mathcal{O}}P(t) = \sum_{0 \leq l} p_{2l+1} t^{2l+1}. \tag{7.5}$$

Theorem (Hermite-Biehler). *Let P be a polynomial, $A = {}^{\mathcal{R}}P$ and $B = {}^{\mathcal{I}}P$ be the real and imaginary parts of P, i.e.*

$$P(t) = A(t) + iB(t),$$

where A and B be a polynomials with real coefficients. In order for all roots of P to be contained within the open upper half plane $\{z : \operatorname{Im} z > 0\}$, it is necessary and sufficient that the following three conditions be satisfied:

1. *The roots of each of the polynomials A and B are all real and simple.*
2. *The sets $\mathcal{Z}_A$ and $\mathcal{Z}_B$ of the roots of the polynomials A and B interlace.*
3. *The inequality*

$$B'(0)A(0) - A'(0)B(0) > 0 \tag{7.6}$$

holds.

Let us formulate a version of Hermite-Biehler Theorem for the left half plane.

Lemma 7.3. *Let M be a polynomial with positive coefficients,*

$$M(t) = \sum_{0 \leq k \leq n} s_k t^k, \quad s_k > 0,\ 0 \leq k \leq n\,,$$

and let $^{\mathcal{E}}M$ and $^{\mathcal{O}}M$ be the even and the odd parts of M. In order for the polynomial M to be dissipative it is necessary and sufficient that the following two conditions be satisfied:

1. *The polynomials $^{\mathcal{E}}M$ and $^{\mathcal{O}}M$ are conservative.*
2. *The sets of roots for the polynomials $^{\mathcal{E}}M$ and $^{\mathcal{O}}M$ interlace.*

Lemma 7.4. *Let W,*

$$W(t) = w_0 + w_2 t^2 + w_4 t^4 \,\cdots + w_{2m-2} t^{2m-2} + w_{2m} t^{2m} \tag{7.7}$$

be an even polynomial with positive coefficients w_{2l}:

$$w_0 > 0,\, w_2 > 0,\, \dots\,,\, w_{2m} > 0\,.$$

In order for the polynomial W to be conservative it is necessary and sufficient that the polynomial $M = W + W'$ be dissipative, where W' is the derivative of W:

$$W'(t) = 2 \cdot w_2 t + 4 \cdot w_4 t^3 \,\cdots + (2m-2) \cdot w_{2m-2} t^{2m-3} + 2m \cdot w_{2m} t^{2m-1}\,. \tag{7.8}$$

Proof of Lemma 7.3. Let

$$P(t)=M(it),\quad A(t)=({}^{\mathcal{E}}M)(it),\quad B(t)=i^{-1}\cdot({}^{\mathcal{O}}M)(it),\tag{7.9}$$

so that

$$P(t)=A(t)+iB(t)\,.$$

A and B are polynomials with real coefficients:

$$A(t)=\sum_{0\le l\le\left[\frac{n}{2}\right]}(-1)^l s_{2l}t^{2l},\quad B(t)=t\sum_{0\le l\le\left[\frac{n-1}{2}\right]}(-1)^l s_{2l+1}t^{2l}\,.$$

Moreover,

$$B'(0)A(0)-A'(0)B(0)=s_0\,s_1\,.\tag{7.10}$$

From (7.9) it is evident that

$$\big(\text{All roots of } A \text{ are real and simple}\big)\Leftrightarrow\big(\text{The polynomial } {}^{\mathcal{E}}M \text{ is conservative}\big)$$
$$\big(\text{All roots of } B \text{ are real and simple}\big)\Leftrightarrow\big(\text{The polynomial } {}^{\mathcal{O}}M \text{ is conservative}\big)$$
$$\big(\text{All roots of } P \text{ lie in } \{z\colon \operatorname{Im} z>0\}\big)\Leftrightarrow\big(\text{The polynomial } M \text{ is dissipative}\big)$$

and under the condition that all roots of A and B are real,

$$\big(\text{The roots of } A \text{ and } B \text{ interlace}\big)\Leftrightarrow\big(\text{The roots of } {}^{\mathcal{E}}M \text{ and } {}^{\mathcal{O}}M \text{ interlace}\big)$$

Thus, Lemma 7.3 is an immediate consequence of the Hermite-Biehler Theorem. *The inequality* (7.6) *is ensured automatically by* (7.10)*, since the coefficients s_k are assumed to be positive.*

□

Proof of Lemma 7.4. It is clear that the polynomials W and W' are, respectively, the even and the odd parts of $M=W+W'$:

$$W={}^{\mathcal{E}}M,\quad W'={}^{\mathcal{O}}M.$$

Let M be dissipative. Then, according to Lemma 7.3, W is conservative. Conversely, let W be conservative. According to Rolle's Theorem, the polynomial W' is also conservative, and the sets of roots belonging to W and W' interlace. By Lemma 7.3, the polynomial M is dissipative. □

Proof of Theorem 2.3. The relation (1.34) means that the polynomial $tW^1_{\partial V}(t)$ is the odd part of the Steiner polynomial S_V. Thus, we are in the situation of Lemma 7.3. Since the polynomial S_V is dissipative, the point $z=0$ is not a root of M, that is, $s_0(V)\neq 0$. According to (1.23), this means that $\mathrm{Vol}_n(V)\neq 0$. Thus, the set V is solid. By Proposition 8.1, all coefficients $s_k(V)$ of the polynomial S_V are strictly positive. According to Lemma 7.3, the polynomial ${}^{\mathcal{O}}(S_V)$ is conservative. Since ${}^{\mathcal{O}}(S_V)(0)=0$, the polynomial $t^{-1}\cdot{}^{\mathcal{O}}(S_V)(t)=W^1_{\partial V}(t)$ is conservative as well. □

Proof of Theorem 5.6. In the course of the proof we shall refer to some facts from the theory of entire functions which are usually formulated in literature for functions whose roots are in the upper rather than in the left half plane. Therefore, it is convenient to pass from the variable t to the variable it. Given a function $H(t)$ of the Hurwitz class $\mathcal{H}$, let $f(t) = H(it)$. Then f is an entire function of exponential type. All roots of f are in the upper half plane and the defect d_f of f is non-negative, where

$$2\, d_f = \overline{\lim_{r\to+\infty}} f(-ir) - \overline{\lim_{r\to+\infty}} f(-ir)\,.$$

(It is clear that $d_f = d_H$, where d_H is the same as in (5.1).) Thus the function f is in the class P as this class was defined in [Lev1, Chapter VII, Section 4]. Let

$$f(t) = A(t) + iB(t)\,,$$

where A and B are real entire functions. Combining Lemma 1 from [Lev1, Chapter VII, Section 4] with Theorem 4 from [Lev1, Chapter VII, Section 2], we see that the functions A and B possess the following properties:

1. A and B are real entire functions of exponential type;
2. $A(0)B'(0) - B(0)A'(0) > 0$;
3. For every $\theta \in \mathbb{R}$, all roots of the linear combination C_θ, where $C_\theta(t) = \cos\theta A(t) + \sin\theta B(t)$, are simple and real. (The entire functions A and B form *a real pair* in the terminology of N.G.Chebotarev, [Cheb].)

According to Hadamard's Factorization Theorem, the entire function C_θ is in the Laguerre-Pólya class. According to the Jensen-Csordas-Williamson Theorem (Theorem 5.5), for each n, all roots of the Jensen polynomial $C_{\theta,n}(t) = \mathfrak{J}_n(C_\theta; t)$ are real and simple. Thus, the real polynomials $A_n(t) = \mathfrak{J}_n(A; t)$ and $B_n(t) = \mathfrak{J}_n(B; t)$ possess the following property: *For every* $\theta \in \mathbb{R}$*, all roots of the linear combination* $\cos\theta A_n(t) + \sin\theta B_n(t)$*, are real and simple.* (The polynomials A_n and B_n are a real pair as well.) From the real pair property of the polynomials A_n and B_n and the property $A_n(0)B_n'(0) - B_n(0)A_n'(0)$ it follows that all roots of the polynomial $f_n(t) = A_n(t) + iB_n(t)$ are in the upper half plane. Thus, all roots of the polynomial $H_n(t) = f_n(-it)$ are in the left half plane. In other words, the polynomial H_n is a Hurwitz polynomial. On the other hand, from the construction it follows that $H_n(t) = \mathfrak{J}_n(H; t)$. □

8. Properties of Steiner polynomials.

RIGID MOTION INVARIANCE: *Let* V*,* $V \subset \mathbb{R}^n$*, be a compact convex set,* τ *be a motion*[23] *of the space* $\mathbb{R}^n$, and τ(V) be the image of the set V under he motion τ. Then $S_{\tau(V)}(t) = S_V(t)$.

CONTINUITY: *The correspondence* $V \to S_V$ *between compact convex sets* V *in* $\mathbb{R}^n$

[23]The rigid motion of the space $\mathbb{R}^n$ is an affine transformation of $\mathbb{R}^n$ which preserves the Euclidean distance in $\mathbb{R}^n$.

and their Steiner polynomials S_V is continuous [24].

A sketch of the proof of the continuity property can be found in [BoFe, section 29], [BuZa, section 19.2], [Schn, section 5.1], [Web].

MONOTONICITY: *Let V_1 and V_2 be compact convex sets in $\mathbb{R}^n$, and S_{V_1}, S_{V_2} be the associated Steiner polynomials. If $V_1 \subset V_2$, then the coefficients $s_k(V_1)$, $s_k(V_2)$ of the polynomials S_{V_1}, S_{V_2}, defined as in* (1.21)*, satisfy the inequalities*

$$s_k(V_1) \leq s_k(V_2), \quad 0 \leq k \leq n\,. \tag{8.1}$$

Explanation. According to the definition of the *mixed volumes*,

$$s_k(V) = \frac{n!}{(n-k)!\,k!}\,\mathrm{Vol}(\underbrace{V, V, \,\ldots\,, V}_{n-k};\,\underbrace{B^n, B^n, \,\ldots\,, B^n}_{k}). \tag{8.2}$$

Inequalities (8.1) follow from the *monotonicity of the mixed volumes* (8.2) *with respect to V*. (Concerning the monotonicity of the mixed volumes see, for example, [BoFe, section 29], [BuZa, section 19.2], [Web, Theorem 6.4.11], [Schn, section 5.1, formula (5.1.23)].)

Lemma 8.1. a). *For any compact convex set V, $V \subset \mathbb{R}^n$, the coefficients $s_k(V)$ of its Steiner polynomial, defined as in* (1.21)*, are non-negative:*

$$0 \leq s_k(V), \quad 0 \leq k \leq n. \tag{8.3a}$$

(*According to* (1.23)*, the coefficient $s_n(V)$ is strictly positive.*)

b). *If, moreover, the set V is solid, then all coefficients $s_k(V)$ are strictly positive:*

$$0 < s_k(V), \quad 0 \leq k \leq n. \tag{8.3b}$$

The Weyl coefficients $w_{2l}(\partial V)$, $0 \leq l \leq \left[\frac{n-1}{2}\right]$, defined by Definition 1.20, *are strictly positive as well.*

Proof. Taking V as V_2 and any one-point subset of V as V_1 in (8.1), we obtain (8.3a). If the set V is solid, then there exist a ball $x_0 + \rho B^n$ of some positive radius ρ: $x_0 + \rho B^n \subset V$. Taking the ball $x_0 + \rho B^n$ as V_1 and V as V_2 in (8.1), we obtain the inequalities $s_k(x_0 + \rho B^n) \leq s_k(V)$, $0 \leq k \leq n$. Moreover, $s_k(x_0 + \rho B^n) = s_k(\rho B^n) = \rho^{n-k} s_k(B^n) = \rho^{n-k} \frac{n!}{k!\,(n-k)!}\,\mathrm{Vol}_n(B^n) > 0$. □

Remark 8.2. The notion of an interior point for a set V depend on the space in which V is embedded. The set V, $V \subset \mathbb{R}^n$, which is non-solid with respect to the 'original' space $\mathbb{R}^n$, is solid if V is considered as embedded in the space $\mathbb{R}^d$ of appropriate dimension d, $d < n$. The dimension $\dim V$ of the set V should be chosen as d.

Definition 8.3. Let V, $V \subseteq \mathbb{R}^n$, be a convex set. The dimension $\dim V$ of V is the dimension of the smallest affine subspace of $\mathbb{R}^n$ which contains V.

[24] The set of compact convex sets in $\mathbb{R}^n$ is equipped with the Hausdorff metric, the set of all polynomials is equipped with the topology of the locally uniform convergence in $\mathbb{C}$.

Lemma 8.4. *Let V, $V \subset \mathbb{R}^n$, be a compact convex set of dimension d:*

$$\dim V = d, \quad 0 \le d \le n. \tag{8.4}$$

Then

$$s_k^{\mathbb{R}^n}(V) = 0 \;\text{ for }\; 0 \le k < n-d; \;\; s_k^{\mathbb{R}^n}(V) > 0 \;\text{ for }\; n-d \le k \le n\,. \tag{8.5}$$

This lemma is a consequence of Lemma 8.1 and of the following

Lemma 8.5. *Let V, $V \subset \mathbb{R}^n$, be a convex set of dimension d, $d \le n$, and let $S_V^{\mathbb{R}^n}(t) = \sum\limits_{0\le k\le n} s_k^{\mathbb{R}^n}(V)t^k$ and*[25] *$S_V^{\mathbb{R}^d}(t) = \sum\limits_{0\le k\le d} s_k^{\mathbb{R}^d}(V)t^k$ be the Minkovski polynomials of the set V with respect to the ambient spaces $\mathbb{R}^n$ and $\mathbb{R}^d$ respectively. Then*

$$S_V^{\mathbb{R}^n}(t) = t^{n-d}\cdot \sum_{0\le k\le d} \frac{\pi^{\frac{n-d}{2}}\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+n-d}{2}+1)} s_k^{\mathbb{R}^d}(V)t^k\,. \tag{8.6}$$

Lemma 8.5 appears in slightly different notation as Theorem 11.3 in Section 11, where a proof is presented.

Definition 8.6. The mixed volumes appearing in (8.2) are said to be cross-sectional measures of the set V and are denoted as $s_{n-k}(V)$:

$$\mathrm{Vol}(\underbrace{V, V, \ldots, V}_{n-k};\, \underbrace{B^n, B^n, \ldots, B^n}_{k}) = s_{n-k}(V), \quad 0 \le k \le n\,. \tag{8.7}$$

Thus, the coefficients of the Minkovski polynomials S_V, which appear in (1.21), can be written as

$$s_k(V) = \binom{n}{k} s_{n-k}(V), \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \text{ are binomial coefficients,} \tag{8.8}$$

and the Steiner polynomial itself can be written as

$$S_V(t) = \sum_{0\le k\le n} \binom{n}{k} v_{n-k}(V)t^k, \tag{8.9}$$

The following fact will be used in Section 10:

Alexandrov − Fenchel Inequalities. *Let V, $V \subset \mathbb{R}^n$, be a compact convex set. Then its cross-sectional measures $v_k(V)$ satisfy the inequalities*

$$v_k^2(V) \ge v_{k-1}(V)\, v_{k+1}(V), \quad 1 \le k \le n-1\,. \tag{8.10}$$

A.D. Alexandrov published two proofs of this inequality in [Al1] and [Al2]. The first of them, a combinatorial proof, is carried out for convex polyhedra. The second proof is more analytical. It uses the theory of self-adjoint elliptic operators depending on a parameter. This proof is carried out for smooth convex sets. To

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[25]Defining the Steiner polynomial $S_V^{\mathbb{R}^d}$, we can assume that the smallest affine subspace of $\mathbb{R}^n$ which contains V is the space $\mathbb{R}^d$.

the general case, both proofs are generalized by limit arguments. The first proof is developed in detail in the textbook [Le]. The second proof is reproduced in Busemann [Bus]. It has become customary to refer to (8.10) as the 'Alexandrov-Fenchel inequality', because Fenchel [Fen] also stated the inequality and sketched the proof. Its detailed presentation was never published. At the end of 1978 Tessier in Paris and A.G.Khovanskiĭ in Moscow both independently obtained an algebraic-geometrical proof of the Alexandrov-Fenchel inequality using the Hodge index theorem. This proof is developed in §27 of the English translation of [BuZa] and was written by A.G. Khovanskiĭ. (In the Russian original of [BuZa] an erroneous algebraic proof of the Alexandrov-Fenchel inequality was included, which has been excluded in the English translation.) Regarding the Alexandrov-Fenchel inequality, see also [BuZa, § 20], and [Schn, Section **6.3**].

Definition 8.7. A sequence $\{p_k\}_{0\leq k\leq n}$ of non-negative numbers:

$$p_k \geq 0\,, \quad 0 \leq k \leq n, \tag{8.11}$$

is said to be logarithmically concave, if the following inequalities hold:

$$p_k^2 \geq p_{k-1}p_{k+1}, \quad 1 \leq k \leq n-1\,. \tag{8.12}$$

Thus, the Alexandrov-Fenchel inequalities can be formulated in the form:

For any convex set V, the sequence $\{v_k(V)\}_{0\leq k\leq n}$ of its cross sectional measures is logarithmically concave.

Under the extra condition (8.11), the logarithmic concavity inequalities (8.12) for the coefficients of the polynomial

$$P(t) = \sum_{0\leq k\leq n} \binom{n}{k} p_k t^k, \tag{8.13}$$

or for the coefficients of the entire function

$$P(t) = \sum_{0\leq k<\infty} \frac{p_k}{k!}\, t^k, \tag{8.14}$$

have been considered in connection with the distribution of the roots belonging to P. In this setting, such (and analogous) inequalities are commonly known as Turán Inequalities (Turán-like Inequalities). Concerning Turán inequalities see, for example, [KaSz] and [CrCs2].

Remark 8.8. The Turán inequalities (8.12) for the coefficients of the polynomial (8.13) or entire function (8.14) impose some restrictions on roots of P. However, these inequalities alone do not ensure that all roots of P are located in the left half plane $\{z : \operatorname{Re} z < 0\}$.

For example, given $m \in \mathbb{N}$, let

$$p_k = 1 \ \text{ for } \ k = 0, 1, \ldots, m \ \text{ and } \ p_k = 0 \ \text{ for } \ k > m. \tag{8.15}$$

Such p_k satisfy the Turán inequalities (8.12). The function (8.14) corresponding to these p_k is the polynomial

$$P_m(t) = \sum_{0 \le k \le m} \frac{t^k}{k!} \,. \tag{8.16}$$

This polynomial is a truncation of the exponential series. It is known that already for $m = 5$ the polynomial (8.16) has two roots located in the half plane $\{z : \operatorname{Re} z > 0\}$. G. Szegö, [Sz], studied the limiting distribution of roots for sequence of polynomials P_m, (8.16), as $m \to \infty$. From his results on the limiting distributions of the roots, it follows that for large m, the polynomial P_m not only has roots in the half plane $\{z : \operatorname{Re} z > 0\}$, but that the total number of its roots located there has a positive density as $m \to \infty$. For further information on the roots of truncated power series we refer the reader to the book [ESV] and to the survey [Ost1]. For $m < n$, the polynomial (8.13) with p_k as in (8.15) takes the form

$$P_{m,n}(t) = \sum_{0 \le k \le m} \binom{n}{k} t^k \,. \tag{8.17}$$

I.V. Ostrovskii, [Ost2], studied the limiting distribution of roots for sequence of the polynomials $P_{m,n}$ as $m, n \to \infty$, $m/n \to \alpha$, $\alpha \in (0,1)$. From his results it follows that for large m, n: $n/m = O(1)$, $n/(n-m) = O(1)$ the polynomial $P_{m,n}$ not only has roots in the half plane $\operatorname{Re} z > 0$, but that the total number of its roots located there has a positive density as $m, n \to \infty$, $m/n \to \alpha \in (0,1)$.

9. The Routh-Hurwitz criterion.

If possible, a geometric approach to finding the location of the roots belonging to Steiner and Weyl polynomials would be useful. At the moment we are, however, not able to do this. The only general tool from geometry which we can use are the the Alexandrov-Fenchel inequalities (8.10) for cross-sectional measures $v_k(V)$ of convex sets. Therefore, one should express all polynomials which we investigate in terms of these cross-sectional measures.

As it was explained in (8.9), the expression of the Steiner polynomial $S_V^{\mathbb{R}^n}$ for the convex set V, $V \subset \mathbb{R}^n$, in terms of the cross-sectional measures $v_k(V)$ is

$$S_V^{\mathbb{R}^n}(t) = \sum_{0 \le k \le n} \binom{n}{k} v_{n-k}(V) t^k \,. \tag{9.1}$$

Lemma 9.1. *Let $\mathcal{M}$ be a closed convex surface,* $\dim \mathcal{M} = n$, *and let V, $V \subset \mathbb{R}^{n+1}$, be a generating convex set: $\mathcal{M} = \partial V$.*

Then the Weyl polynomial $W_{\mathcal{M}}^{\infty}$ can be expressed as

$$W_{\mathcal{M}}^{\infty}(t) = \sum_{0 \le l \le \left[\frac{n}{2}\right]} \frac{(n+1)!}{2^l l! (n-2l)!} \, v_{n-2l}(V) \, t^{2l} \,, \tag{9.2}$$

where $v_k(V)$ are the cross-sectional measures of the generating convex set V.

Proof. The expression (9.2) is a consequence of (1.41), (1.38) and (8.8). $\square$

The inequalities (8.10) for the coefficients of the polynomials (9.1) and (9.2) comprise one of two general tools, which will be used in the study of the location of roots of these polynomials. The second general tool at our disposal is the collection of criteria describing the dissipativity (or, alternatively conservativeness) property of polynomials in term of their coefficients.

Theorem (Routh-Hurwitz). *Let*

$$P(t) = a_0 t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_n \tag{9.3}$$

be a polynomial with strictly positive coefficients:

$$a_0 > 0,\, a_1 > 0,\, \ldots ,\, a_{n-1} > 0,\, a_n > 0\,. \tag{9.4}$$

For the polynomial P to be dissipative, it is necessary and sufficient that all the determinants Δ_k, $k = 1, 2, \ldots , n-1, n$, be strictly positive:

$$\Delta_1 > 0,\, \Delta_2 > 0,\, \ldots ,\, \Delta_{n-1} > 0,\, \Delta_n > 0, \tag{9.5}$$

where

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix},$$

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}, \quad \ldots \quad , \Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \ldots & 0 \\ a_0 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & a_n \end{vmatrix}. \tag{9.6}$$

This result, as well as many related results, can be found in [Gant, Chapter XV]. See also [KrNa].

Remark 9.2. Actually, to prove that the polynomial P, (9.3), of degree n with positive coefficients a_k, $k = 1, 2, \ldots , n$, is dissipative, there is no need to inspect *all* Hurwitz determinants Δ_k, $k = 1, 2, \ldots , n$, for positivity. It is enough to inspect either the determinants Δ_k with even k, or the determinants Δ_k with odd k. (See [Gant, Chapter XV, §13].)

Applying the Routh-Hurwitz criterion to determine whether the Steiner polynomial $S_V^{\mathbb{R}^n}$ is dissipative, we should take, according to (9.1),

$$a_k = \frac{n!}{k!(n-k)!} v_k(V)\,, \quad 0 \leq k \leq n, \quad a_k = 0, \quad k > n\,. \tag{9.7}$$

From the criterion for dissipativity, the criterion of conservativeness can be derived easily.

Theorem (Criterion for conservativeness). *Let*

$$P(t) = a_0 t^{2m} + a_2 t^{2m-2} + \dots + a_{2m-2} t^2 + a_{2m} \tag{9.8}$$

be a polynomial with strictly positive coefficients a_{2l}, $0 \le l \le m$*:*

$$a_0 > 0,\, a_2 > 0,\, \dots,\, a_{2m-2} > 0,\, a_{2m} > 0\,. \tag{9.9}$$

For the polynomial P to be conservative, it is necessary and sufficient that all the determinants D_k, $k = 1, 2, \dots, 2m-1, 2m$, be strictly positive:

$$D_1 > 0,\, D_2 > 0,\, D_3 > 0,\, \dots,\, D_{2m-1} > 0,\, D_{2m} > 0, \tag{9.10}$$

where D_k are constructed from the coefficients of the polynomial P according to the following rule:D_k is the determinant Δ_k, (9.6)*, whose entries a_{2l}, $0 \le l \le m$, are the coefficients of the polynomial P, and $a_{2l+1} = (m-l)\, a_{2l}$, $0 \le l \le m-1$:*

$$D_1 = m\, a_0, \quad D_2 = \begin{vmatrix} m\, a_0 & (m-1)a_2 \\ a_0 & a_2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} m a_0 & (m-1)a_2 & (m-2)a_4 \\ a_0 & a_2 & a_4 \\ 0 & m\, a_0 & (m-1)a_2 \end{vmatrix},$$

$$D_4 = \begin{vmatrix} m\, a_0 & (m-1)\, a_2 & (m-2)a_4 & (m-4)a_6 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & m\, a_0 & (m-1)\, a_2 & (m-2)a_4 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}, \quad \dots \tag{9.11}$$

$$D_{2m} = \begin{vmatrix} m\, a_0 & (m-1)a_2 & (m-2)a_4 & \dots & 0 \\ a_0 & a_2 & a_4 & \dots & 0 \\ 0 & m a_0 & (m-1)a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & a_{2m} \end{vmatrix}.$$

Proof. This theorem is the immediate consequence of the Hermite-Biehler Theorem and Lemma 7.4. $\square$

Applying criterion for conservativeness to determine whether the Weyl polynomial $W^{\infty}_{\mathcal{M}}$ is conservative, we should take, according to (9.2) and (9.8),

$$a_{2l} = \frac{(n+1)!}{2^{m-l}(m-l)!(2l+n-2m)!} v_{2l+n-2m}(V)\,,$$
$$0 \le l \le m, \quad a_{2l} = 0, \quad l > m\,, \quad \text{where } m = \left[\tfrac{n}{2}\right] \tag{9.12}$$

10. The case of low dimension: proofs of Theorems 2.5 and 2.6.

Proof of Theorem 2.5. We apply the Routh-Hurwitz criterion for dissipativity to the Steiner polynomial $S^{\mathbb{R}^n}_V$. 'Expanding' the Hurwitz determinants Δ_k, (9.6) , and

taking into account that $a_k = 0$ for $k > n$, we obtain that for $n \le 5$,

$$\Delta_1 = a_1 \tag{10.1.1}$$

$$\Delta_2 = a_1a_2 - a_0a_3\,, \tag{10.1.2}$$

$$\Delta_3 = a_1a_2a_3 + a_0a_1a_5 - a_0a_3^2 - a_1^2a_4\,, \tag{10.1.3}$$

$$\Delta_4 = a_1a_2a_3a_4 + a_0a_2a_3a_5 + 2a_0a_1a_4a_5 - a_1^2a_4^2 - a_0^2a_5^2 - a_0a_3^2a_4 - a_1a_2^2a_5, \tag{10.1.4}$$

$$\Delta_5 = a_5\Delta_4\,, \tag{10.1.5}$$

where we should take a_k as in (9.7).

According to the Routh-Hurwitz criterion, we have to prove that $\Delta_1 > 0$, $\Delta_2 > 0, \ldots, \Delta_n > 0$. The cases $n = 2, 3, 4, 5$ will be considered separately. Since V is solid, we have that $v_k(V) > 0$, $0 \le k \le n$. (Corollary 8.1.b and (8.8).) Thus, the determinant $\Delta_1 = \binom{n}{1}v_1(V)$ is always positive.

The cases $n = 2, 3, 4, 5$ will again be considered separately. For simplicity, we write v_k instead $v_k(V)$.

***n* = 2.** In this case,

$$a_0 = v_0,\ a_1 = 2v_1,\ a_2 = v_2,$$

$$\Delta_2 = 2v_2v_1\,.$$

The inequality $\Delta_2 > 0$ is evident. Thus, the Steiner polynomial $S_V^{\mathbb{R}^2}$ is dissipative.

***n* = 3.** In this case,

$$a_0 = v_0, \quad a_1 = 3v_1, \quad a_2 = 3v_2, \quad a_3 = v_3\,, \quad a_k = 0,\ k > 3\,.$$

Substituting these expressions for a_k into (10.1), we obtain

$$\Delta_2 = 9v_1v_2 - v_0v_3, \quad \Delta_3 = v_3\Delta_2\,.$$

Thus, $S_V^{\mathbb{R}^3}$ is dissipative if, and only if

$$9v_1v_2 > v_0v_3, \tag{10.2}$$

***n* = 4.** In this case,

$$a_0 = v_0, \quad a_1 = 4v_1, \quad a_2 = 6v_2, \quad a_3 = 4v_3 \quad a_4 = v_4, \quad a_k = 0,\ k > 4\,.$$

Substituting these expressions for a_k into (10.1), we obtain

$$\Delta_2 = 24v_1v_2 - 4v_0v_3, \quad \Delta_3 = 96v_1v_2v_3 - 16v_0v_3^2 - 16v_1^2v_4\,, \quad \Delta_4 = v_4\Delta_3\,.$$

Thus, $S_V^{\mathbb{R}^4}$ is dissipative if, and only if the pair of inequalities

$$6v_1v_2 > v_0v_3, \tag{10.3a}$$

$$6v_1v_2v_3 > v_0v_3^2 + v_1^2v_4\,. \tag{10.3b}$$

***n* = 5.** In this case,

$$a_0 = v_0,\ a_1 = 5v_1,\ a_2 = 10v_2,\ a_3 = 10v_3,\ a_4 = 5v_4,\ a_5 = v_5,\ a_k = 0,\ k > 5\,.$$

Substituting these expressions for a_k into (10.1), we obtain

$$\Delta_2 = 50v_1v_2 - 10v_0v_3, \quad \Delta_3 = 500v_1v_2v_3 + 5v_0v_1v_5 - 100v_0v_3^2 - 125v_1^2v_4\,,$$
$$\Delta_4 = 2500v_1v_2v_3v_4 + 100v_0v_2v_3v_5 + 50v_0v_1v_4v_5$$
$$- 625v_1^2v_4^2 - v_0^2v_5^2 - 500v_0v_3^2v_4 - 500v_0v_1^2v_5\,, \quad \Delta_5 = v_5\Delta_4\,.$$

Thus, $S_V^{\mathbb{R}^5}$ is dissipative iff the following three inequalities hold:

$$5v_1v_2 > v_0v_3, \tag{10.4a}$$
$$100v_1v_2v_3 + v_0v_1v_5 > 20v_0v_3^2 + 25v_1^2v_4\,, \tag{10.4b}$$
$$2500v_1v_2v_3v_4 + 100v_0v_2v_3v_5 + 50v_0v_2v_4v_5 > \tag{10.4c}$$
$$> 625v_1^2v_4^2 + 500v_0v_3^2v_4 + 500v_1v_2^2v_5 + v_0^2v_5^2\,.$$

As it is claimed below in Lemma 10.2, the inequalities (10.2), (10.3), (10.4), where $v_k = v_k(V)$ are the cross-sectional measures of the solid compact set V of appropriate dimension, are consequences of the Alexandrov-Fenchel inequalities. This completes the proof. $\square$

Remark 10.1. The above ***does not*** hold for all n. If n is large enough, then the conditions $v_k^2 \geq v_{k-1}v_{k+1}$, $\;1 \leq k \leq n-1$, for positive numbers v_k, do not imply the inequalities $\Delta_k \geq 0$ for all $k = 1, \dots, n$, where Δ_k is constructed from a_k's given by $a_k = \binom{n}{k}v_k$. Already for $n = 30$, $\Delta_5 < 0$ for certain v_k satisfying these conditions. Moreover, as we will see later, for n large enough, there exist examples of compact convex sets $V \subset \mathbb{R}^n$ for which the Steiner polynomial S_V is not dissipative and the Weyl polynomial $W^1_{\partial V}$ is not conservative. Although the sets V in such examples are solid, they are "almost degenerate".

Lemma 10.2. *Let v_k, $0 \leq k \leq n$, be strictly positive numbers satisfying the inequalities*

$$v_k^2 \geq v_{k-1}v_{k+1}, \quad 1 \leq k \leq n-1\,. \tag{10.5}$$

Then:

1. *If $n = 3$, the inequality* (10.2) *holds;*
2. *If $n = 4$, the inequalities* (10.3) *holds;*
3. *If $n = 5$, the inequalities* (10.4) *holds.*

Proof of Lemma 10.2. The *logarithmic concavity* inequalities (10.5) imply the inequalities

$$v_pv_s \geq v_qv_r\,, \tag{10.6}$$

where p, q, r, s are arbitrary indices satisfying the conditions

$$1 \leq p \leq q \leq r \leq s \leq n\,, \quad p + s = q + r\,. \tag{10.7}$$

$\boldsymbol{n = 3}$. $\qquad v_1v_2 \geq v_0v_3 \quad \Rightarrow \quad (10.2)$

$\boldsymbol{n = 4}$. $\qquad v_1v_2 \geq v_0v_3\,, \quad v_2v_3 \geq v_1v_4 \quad \Rightarrow \quad (10.3)$

$\boldsymbol{n=5}$.

$$v_1v_2 \geq v_0v_3\,,\ v_0v_5 \geq v_1v_4\,,\ v_2v_3 \geq v_1v_4\,,\ v_3v_4 \geq v_2v_5\,,\quad v_2v_3 \geq v_0v_5 \quad\Rightarrow \tag{10.4}$$

□

PROOF of THEOREM 2.6. We apply the Conservativeness Criterion, which was formulated in the previous section, for Weyl polynomial $W^{\infty}_{\mathcal{M}}$. Expanding the determinants D_k, (9.11) , and taking into account that $a_{2l}=0$ for $l > \left[\frac{n}{2}\right]$, we obtain that for $n \leq 5$, that is,[26] for $m = \left[\frac{n}{2}\right] \leq 2$,

$$D_1 = ma_0 \tag{10.8.1}$$

$$D_2 = a_0a_2\,, \tag{10.8.2}$$

$$D_3 = a_0\big((m-1)a_2^2 - 2ma_0a_4\big)\,, \tag{10.8.3}$$

$$D_4 = a_0a_4(a_2^2 - 4a_0a_4)\,. \tag{10.8.4}$$

where we take a_{2l} as in (9.12).

According to the Conservativeness Criterion, we have to prove that $D_1 > 0$, $D_2 > 0$, ... , $D_{2m} > 0$, where $m = \left[\frac{n}{2}\right]$.
Since V is solid, $v_k(V) > 0$, $0 \leq k \leq n+1$. (Corollary 8.1.b and (8.8) .) Thus, the determinants D_1, D_2 are always positive.

Therefore, if $n=2$ or if $n=3$, that is, if $m=1$, the Weyl polynomial $W^{\infty}_{\mathcal{M}}$ is conservative. Of course, this fact is evident without referring to the conservativeness criterion:

In the case $\boldsymbol{n=2}$, according to (9.7) or (9.2),

$$W^{\infty}_{\mathcal{M}}(t) = 3v_2 + 3v_0t^2\,.$$

In the case $\boldsymbol{n=3}$, according to (9.7) or (9.2),

$$W^{\infty}_{\mathcal{M}}(t) = v_3 + 3v_1t^2\,.$$

In both cases, $n=2$ or $n=3$, the polynomial $W^{\infty}_{\mathcal{M}}$ is clearly conservative.

In the cases $n=4,\ n=5$, which corresponds to the case where $m=2$,

$$D_3 = a_0(a_2^2 - 4a_0a_4), \quad D_4 = a_4D_3\,.$$

According to (9.12), we must also take the following two cases into account:
In the case that $\boldsymbol{n=4}$

$$a_0 = 15v_0, \quad a_2 = 30v_2, \quad a_4 = 5v_4\,.$$

Thus,

$$D_3 = 15v_0(900v_2^2 - 300v_0v_4)\,.$$

The conditions $D_3 > 0$, $D_4 > 0$ take the form

$$3v_2^2 > v_0v_4\,. \tag{10.9}$$

In the case that $\boldsymbol{n=5}$

$$a_0 = 90v_1, \quad a_2 = 60v_3, \quad a_4 = 6v_5\,.$$

[26] Recall that $n = \dim \mathcal{M}$, $n+1 = \dim V : \mathcal{M} = \partial V$.

Thus,

$$D_3 = 90v_1(900v_2^2 - 300v_0v_4)\,.$$

The conditions $D_3 > 0$, $D_4 > 0$ take the form

$$5v_3^2 > 3v_1v_5\,. \tag{10.10}$$

The Weyl polynomial $W^{\infty}_{\mathcal{M}}$ is, therefore, conservative in the cases $n = 4$ and $n = 5$ if the inequalities (10.9) and (10.10) hold, respectively. $v_k = v_k(V)$ are the cross-sectional measures of the solid compact set V generating the surface $\mathcal{M}$: $\mathcal{M} = \partial V$. The inequalities (10.9) and (10.10) are, in turn, consequences of the inequalities $v_2^2 \geq v_0v_4$ and $v_3^2 \geq v_1v_5$, respectively. The latter inequalities are special cases of the inequalities (10.6). Thus, in the cases $n = 4$ and $n = 5$ the Weyl $W^{\infty}_{\mathcal{M}}$ polynomial with infinite index is conservative. By Lemma 2.4, all Weyl polynomials $W^{p}_{\mathcal{M}}$, $p = 1, 2, 3, \dots$, are conservative as well. □

11. Extending the ambient space.

Adjoint convex sets. Let V be a compact convex set, $V \subset \mathbb{R}^n$. We may consider the space $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+q}$, where $q = 1, 2, 3, \dots$. The embedding $\mathbb{R}^n$ in $\mathbb{R}^{n+q}$ is canonical:

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+q}: \quad (\xi_1, \dots, \xi_n) \to (\xi_1, \dots, \xi_n; \underbrace{0, \dots, 0}_{q}).$$

Thus, the set V, which originally was considered as a subset of $\mathbb{R}^n$, may also be considered as a subset of $\mathbb{R}^{n+q}$. In other words, we identify the set $V \subset \mathbb{R}^n$ with the set $V \times 0^q$, which is the Cartesian product of the set V and the origin 0^q of the space $\mathbb{R}^q$: $V \times 0^q \subset \mathbb{R}^{n+q}$.

Definition 11.1. Given a compact convex set V, $V \subset \mathbb{R}^n$, and a number q, $q = 0, 1, 2, 3, \dots$, the q-th adjoint to V set $V^{(q)}$ is defined as:

$$V^{(q)} \stackrel{\text{def}}{=} V \times 0^q, \quad V^{(q)} \subset \mathbb{R}^{n+q}\,, \tag{11.1}$$

where 0^q is the zero point of the space $\mathbb{R}^q$, and the space $\mathbb{R}^{n+q}$ is considered as the Cartesian product: $\mathbb{R}^{n+q} = \mathbb{R}^n \times \mathbb{R}^q$.
The Steiner polynomial $S^{\mathbb{R}^{n+q}}_{V\times 0^q}$ of the q-th adjoint set, denoted by $V^{(q)}$,

$$S^{\mathbb{R}^{n+q}}_{V\times 0^q} = \mathrm{Vol}_{n+q}(V \times 0^q + tB^{n+q}), \tag{11.2}$$

is said to be the q-th adjoint Steiner polynomial for the set V.

For $q = 0$, the set $V^{(0)}$ coincides with V, and the polynomial $S^{\mathbb{R}^{n+0}}_{V\times 0^0}$ coincides with $S^{\mathbb{R}^n}_V$. For $q = 1$, the set $V^{(1)}$ is called the squeezed cylinder with the base V.

Steiner polynomials for adjoint sets. Let us consider the following question:

What is the relationsheep between the polynomials $S_V^{\mathbb{R}^n}(t)$ *and* $S_{V\times 0^q}^{\mathbb{R}^{n+q}}(t)$?

We have

Lemma 11.2. *Let* V *be a compact convex set in* $\mathbb{R}^n$*, and*

$$S_V^{\mathbb{R}^n}(t) = \sum_{0\le k\le n} s_k^{\mathbb{R}^n}(V)t^k \tag{11.3}$$

be the Steiner polynomial with respect to the ambient space $\mathbb{R}^n$*. Then the Steiner polynomial* $S_{V\times 0}^{\mathbb{R}^{n+1}}(t)$ *is equal to:*

$$S_{V\times 0^1}^{\mathbb{R}^{n+1}}(t) = t\sum_{0\le k\le n} \frac{\pi^{1/2}\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\, s_k^{\mathbb{R}^n}(V)\, t^k\,. \tag{11.4}$$

Proceeding the induction over q and using Lemma 11.2 we obtain

Theorem 11.3. *Let* V *be a compact convex set in* $\mathbb{R}^n$*, and*

$$S_V^{\mathbb{R}^n}(t) = \sum_{0\le k\le n} s_k^{\mathbb{R}^n}(V)t^k \tag{11.5}$$

be the Steiner polynomial of the set V*. Then the* q*-th adjoint Steiner polynomial* $S_{V\times 0^q}^{\mathbb{R}^{n+q}}(t)$ *of the set* V *is equal to:*

$$S_{V\times 0^q}^{\mathbb{R}^{n+q}}(t) = \sum_{0\le k\le n} s_k^{\mathbb{R}^n}(V)\,\gamma_k^{(q)}\; t^{k+q}\,, \tag{11.6}$$

where

$$\gamma_k^{(q)} = \pi^{q/2}\,\frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+q}{2}+1)}\,, \quad k=0,\,1,\,2,\,\dots\,; \quad q=0,\,1,\,2,\,\dots\,. \tag{11.7}$$

A sketch of the proof for this theorem can be found in [Had, Chapter VI, Section 6.1.9]. A detailed proof is presented below.

Remark 11.4. Theorem 11.3 means that the sequence of the coefficients $\{s_k^{\mathbb{R}^{n+q}}(V\times 0^q)\}_{0\le k\le n+q}$ of the polynomial $S_{V\times 0^q}^{\mathbb{R}^{n+q}}$:

$$S_{V\times 0^q}^{\mathbb{R}^{n+q}}(t) = \sum_{0\le k\le n+q} s_k^{\mathbb{R}^{n+q}}(V\times 0^q)\,t^k \tag{11.8}$$

are obtained from the sequence of the coefficients $\{s_k(V)\}_{0\le k\le n}$ of the polynomial $S_V^{\mathbb{R}^n}$, (11.5), by means of a shift and multiplication:

$$s_k^{\mathbb{R}^{n+q}}(V\times 0^q) = 0, \qquad 0\le k<q; \tag{11.9a}$$

$$s_{k+q}^{\mathbb{R}^{n+q}}(V\times 0^q) = s_k^{\mathbb{R}^n}(V)\,\gamma_k^{(q)}, \quad 0\le k\le n\,. \tag{11.9b}$$

Remark 11.5. According to Theorem 11.3, the transformation which maps the polynomial $S_V^{\mathbb{R}^n}$ into the polynomial $S_{V\times 0^q}^{\mathbb{R}^{n+q}}$ is of the form

$$\sum_{0\le k\le n} s_k t^k \to \sum_{0\le k\le n} \gamma_k s_k t^k, \tag{11.10}$$

where γ_k is a certain sequence of multipliers. (The factor t^q in front of the sum in (11.6) can be omitted). Transformations of the form (11.10) were already discussed is Section 5. Such transformations were considered in relation with locations of roots for polynomials and entire functions.

Lemma 11.6. *For any q, $q = 1, 2, 3, \ldots$, the sequence $\{\gamma_k^{(q)}\}_{k=0,1,2,\ldots}$ is not a multiplier sequence in the sense of* Definition 5.4.

Proof. In Section 13 we explain that the entire function

$$\mu_q(t) = \sum_{0\le k<\infty} \frac{\gamma_k^{(q)}}{k!} t^k \tag{11.11}$$

has infinitely many non-real roots. The entire function $\mu_q(t)$, (11.11), appears as the function $\mathcal{M}_{B^\infty\times 0^q}(t)$ in Section (11.26). (Up to a constant factor which is not essential for study the roots.) According to the Pólya-Schur Theorem, which was formulated in 5, the sequence $\{\gamma_k^{(q)}\}_{k=0,1,2,\ldots}$ is not a multiplier sequence. □

Remark 11.7. In Section 13 we study the function μ_q in much more detail that is needed to prove Lemma 11.6. We investigate for which q function's roots are located in the left half plane. The question of whether or not there are non-real roots is much less involved. This question may be answered from very general considerations. The function μ_q admits the integral representation:

$$\mu_q(t) = q\omega_q \int\limits_0^1 (1-\xi^2)^{\frac{q}{2}-1} \xi\, e^{\xi t}\, d\xi\,. \tag{11.12}$$

(Expanding the exponential $e^{\xi t}$ into the Taylor series, we see that the Taylor coefficients of the function on the right hand side of (11.12) are the numbers $\frac{\gamma_k^{(q)}}{k!}$.) From (11.12) it follows that the function $\mu_q(t)$ is an entire function of exponential type and that its indicator diagram is the interval [0, 1]. Moreover, $\sup_{-\infty<t<\infty} |\mu_q(it)| < \infty$. In particular, the function $\mu_q(it)$ belongs to the class of entire functions, denoted by C in [Lev2, Lecture 17]. It follows from the Cartwright-Levinson Theorem (Theorem 1 of the Lecture 17 from [Lev2]) that the function $\mu_q(t)$ has infinitely many roots. These roots have positive density, and the 'majority' of these roots are located 'near' the rays $\arg t = \frac{\pi}{2}$ and $\arg t = -\frac{\pi}{2}$. In particular, the function $\mu_q(t)$ has infinitely many non-real roots. (We already used this reasoning to prove Statement 2 of Theorem 6.4.)

Proof of Lemma 11.2. Let $(x,s)\in\mathbb{R}^{n+1}$, where $x\in\mathbb{R}^n$, and $s\in\mathbb{R}$. Then, by the Pythagorean Theorem,

$$\operatorname{dist}^2_{\mathbb{R}^{n+1}}((x,s),V\times 0)=\operatorname{dist}^2_{\mathbb{R}^n}(x,V)+s^2.$$

Therefore, the equivalence

$$\Big(\operatorname{dist}_{\mathbb{R}^{n+1}}((x,s),V\times 0)\leq t\Big)\Longleftrightarrow\Big(\operatorname{dist}_{\mathbb{R}^n}(x,V)\leq\sqrt{t^2-s^2}\Big) \tag{11.13}$$

holds. Let

$$\mathfrak{T}^{\mathbb{R}^{n+1}}_{V\times 0^1}(t)=\{(x,s)\in\mathbb{R}^{n+1}:\,\operatorname{dist}_{\mathbb{R}^{n+1}}((x,s),V\times 0^1)\leq t\}. \tag{11.14}$$

be the t-neighborhood of the set $V\times 0^1$ with respect to the ambient space $\mathbb{R}^{n+1}$. Thus,

$$\operatorname{Vol}_{n+1}(\mathfrak{T}^{\mathbb{R}^{n+1}}_{V\times 0^1}(t))=S^{\mathbb{R}^{n+1}}_{V\times 0^1}(t). \tag{11.15}$$

For fixed $s\in\mathbb{R}$, let $\mathfrak{S}(s)$ be the 'horizontal section' of the set $\mathfrak{T}^{\mathbb{R}^{n+1}}_{V\times 0^1}(t)$ on the 'vertical level' s:

$$\mathfrak{S}(s)=\{x\in\mathbb{R}^n:\,(x,s)\in\mathfrak{T}^{\mathbb{R}^{n+1}}_{V}(t)\}.$$

Clearly

$$\operatorname{Vol}_{n+1}(\mathfrak{T}^{\mathbb{R}^{n+1}}_{V\times 0^1}(t))=\int\operatorname{Vol}_n(\mathfrak{S}(s))ds\,. \tag{11.16}$$

The equivalence (11.13) means that

$$\mathfrak{S}(s)=\mathfrak{T}^{\mathbb{R}^n}_V(\sqrt{t^2-s^2})=\{x\in\mathbb{R}^n:\,\operatorname{dist}_{\mathbb{R}^n}(x,V)\leq\sqrt{t^2-s^2}\}.$$

Thus,

$$\operatorname{Vol}_n(\mathfrak{S}(s))=S^{\mathbb{R}^n}_V(\sqrt{t^2-s^2}). \tag{11.17}$$

From (11.16) and (11.17) it follows that

$$S^{\mathbb{R}^{n+1}}_{V\times 0^1}(t)=\int\limits_{-t}^{t}S^{\mathbb{R}^n}_V(\sqrt{t^2-s^2})ds\,.$$

Making the substitutions $s\to ts^{1/2}$, we obtain

$$S^{\mathbb{R}^{n+1}}_V(t)=t\int\limits_0^1 S^{\mathbb{R}^n}_V(t(1-s)^{1/2})s^{-1/2}ds.$$

Substituting the expression (1.31) for $S^{\mathbb{R}^n}_V$ into the latter formula, we obtain

$$S^{\mathbb{R}^{n+1}}_{V\times 0^1}(t)=t\sum_{0\leq k\leq n}s_k(V)\,t^k\int\limits_0^1(1-s)^{k/2}s^{-1/2}ds.$$

According to Euler,

$$\int\limits_0^1(1-s)^{k/2}s^{-1/2}ds=\mathrm{B}\big(\tfrac{k}{2}+1,\tfrac{1}{2}\big)=\frac{\Gamma\big(\frac{1}{2}\big)\,\Gamma\big(\frac{k}{2}+1\big)}{\Gamma\big(\frac{k+1}{2}+1\big)}=\pi^{1/2}\frac{\Gamma\big(\frac{k}{2}+1\big)}{\Gamma\big(\frac{k+1}{2}+1\big)}\cdot$$

Thus, (11.4) holds. □

Proof of Theorem 11.3. We proceed by induction over q. For $q=0$, the assertion is trivially clear. We now show that if the statement holds for q, then it also holds for $q+1$. Since $V\times 0^{q+1}=(V\times 0^q)\times 0^1$, and $\mathbb{R}^{n+q+1}=\mathbb{R}^{n+q}\times\mathbb{R}^1$, we can apply Lemma 11.2 to the convex set $V\times 0^q$ whose Steiner polynomial is (11.6) (by the inductive hypothesis). The inductive hypothesis can be formulated as

$$s_k^{\mathbb{R}^{n+q}}(V\times 0^q)=0, \qquad 0\le k<q; \tag{11.18a}$$

$$s_k^{\mathbb{R}^{n+q}}(V\times 0^q)=s_{k-q}^{\mathbb{R}^n}(V)\,\gamma_{k-q}^{(q)}, \quad q\le k\le q+n\,. \tag{11.18b}$$

By Lemma 1.15,

$$s_k^{\mathbb{R}^{(n+q)+1}}((V\times 0^q)\times 0^1)=0, \quad k=0\,; \tag{11.19a}$$

$$s_k^{\mathbb{R}^{(n+q)+1}}((V\times 0^q)\times 0^1)=s_{k-1}^{\mathbb{R}^{(n+q)}}((V\times 0^q))\cdot\gamma_{k-1}^{(1)}\,, \quad 1\le k\le n+q+1\,. \tag{11.19b}$$

In view of the identity

$$\gamma_k^{(q)}\cdot\gamma_{k+q}^{(1)}=\gamma_k^{(q+1)}\,,$$

(11.19) takes the form (11.18) with q replaced by $q+1$. □

Remark 11.8. From (1.6) and (11.7) it follows that

$$\gamma_k^{(q)}=\frac{\omega_{k+q}}{\omega_k}\,. \tag{11.20}$$

Thus, the equalities (11.9b) can be rewritten as

$$\frac{s_{k+q}^{\mathbb{R}^{n+q}}(V\times 0^q)}{\omega_{k+q}}=\frac{s_k^{\mathbb{R}^n}(V)}{\omega_k}, \quad q=0,\,1,\,2,\,\dots\,. \tag{11.21}$$

The equality (11.21) holds for $k=0,\,1,\,\dots\,,\,n$. The value $s_k^{\mathbb{R}^n}(V)$ is not yet defined for other values of k. Let us agree that

$$\omega_k=1 \;\text{ for }\; k<0, \quad s_k^{\mathbb{R}^n}(V)=0 \;\text{ for }\; k<0 \;\text{ and for }\; k>n\,. \tag{11.22}$$

The equality (11.21) then holds for *every* $k\in\mathbb{Z}$. For $k>n$ or for $k<-q$ (11.21) is trivial. For $-q\le k\le -1$ (11.21) coincides with (11.9a). For $0\le k\le n$ (11.21) coincides with (11.9b). The ratio $\dfrac{s_{n-k}^{\mathbb{R}^n}(V)}{\omega_{n-k}}$ is called the *k-th intrinsic volume of* V in [McM].

The Steiner polynomials for the q-th adjoint to the ball B^n. In particular, applying Theorem 11.3 to the case $V = B^n$, $B^n \subset \mathbb{R}^n$, we obtain:

$$S^{\mathbb{R}^{n+q}}_{B^n\times 0^q}(t) = \omega_n\omega_q\, t^q \mathcal{M}_{B^n\times 0^q}(nt)\,, \tag{11.23}$$

where the normalized Steiner polynomial $\mathcal{M}_{B^n\times 0^q}$ is defined as

$$\mathcal{M}_{B^n\times 0^q}(t) = \sum_{0\leq k\leq n} \frac{n!}{(n-k)!n^k} \frac{\Gamma(\frac{q}{2}+1)\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+q}{2}+1)} \frac{t^k}{k!}\,. \tag{11.24}$$

The polynomial $\mathcal{M}_{B^n\times 0^q}$ is the Jensen polynomial associated with the entire function $M^{B^n\times 0^q}$:

$$\mathcal{M}_{B^n\times 0^q}(t) = \mathcal{J}_n(\mathcal{M}_{B^\infty\times 0^q};t), \tag{11.25}$$

where

$$\mathcal{M}_{B^\infty\times 0^q}(t) = \sum_{0\leq k<\infty} \frac{\Gamma(\frac{q}{2}+1)\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+q}{2}+1)} \frac{t^k}{k!}\,. \tag{11.26}$$

Comparing (11.26) with (11.12), we obtain

$$\mathcal{M}_{B^\infty\times 0^q}(t) = q\int\limits_0^1 (1-\xi^2)^{\frac{q}{2}-1}\xi\, e^{\xi t}\, d\xi\,. \tag{11.27}$$

For every $q = 0, 1, 2, \ldots$, the function $\mathcal{M}_{B^\infty\times 0^q}$ is an entire function of the exponential type one.

Lemma 11.9.

1. *For $q = 0, 1, 2, 4$, the entire function $\mathcal{M}_{B^\infty\times 0^q}$ is in the Hurwitz class $\mathcal{H}$;*
2. *For $q \geq 5$, the entire function $\mathcal{M}_{B^\infty\times 0^q}$ is not in the Hurwitz class: it has infinitely many roots in the open right half plane $\{z : \operatorname{Im} z > 0\}$.*

Proof of this Lemma is presented in Section 13. Statement 2 is a consequence of the asymptotic calculation of the function $\mathcal{M}_{B^\infty\times 0^q}$. (See Lemma 13.1.)

For $q = 0$, we have that $\mathcal{M}_{B^\infty\times 0^0} = e^t$. $\mathcal{M}_{B^\infty\times 0^0} = e^t$ is therefore of type I in the Laguerre-Pólya class: $\mathcal{M}_{B^\infty\times 0^0} \in \mathcal{L}$-$\mathcal{P}$-I. For $q = 2$ and $q = 4$ the functions $\mathcal{M}_{B^\infty\times 0^q}$ can be calculated explicitly. The case $q = 1$ is more involved. The case $q = 3$ remains open.

Proof of Statement 1 of Theorem 2.12. Let $q \geq 5$ be given. According to Statement 2 of Lemma 11.9, the function $\mathcal{M}_{B^\infty\times 0^q}$ has infinitely many roots in the open right half plane. In view of the approximation property of the Jensen polynomials (Lemma 4.2), some roots of the Jensen polynomial $\mathcal{J}_n(\mathcal{M}_{B^\infty\times 0^q};t)$ for $n \geq n(q)$ are located in the open right half plane. Because of (11.23) and (11.25), some roots of the Minkovski polynomial $S_{B^n\times 0^q}$ of the (non-solid) convex set $B^n \times 0^q$, $B^n \times 0^q \subset \mathbb{R}^{n+q}$, are located in the open right half plane. For fixed $n : n \geq n(q)$, consider the ellipsoids $E_{n,\,q,\,\varepsilon}$ defined in (2.9), $E_{n,\,q,\,\varepsilon} \subset \mathbb{R}^{n+q}$. For $\varepsilon > 0$, the ellipsoid $E_{n,\,q,\,\varepsilon}$ is a solid convex set with respect to the ambient space

$\mathbb{R}^{n+q}$. The family of convex sets $\{E_{n,\,q,\,\varepsilon}\}_{\varepsilon>0}$ is monotonic, (See Remark 1.16 and footnote 7), and

$$\lim_{\varepsilon\to+0} E_{n,\,q,\,\varepsilon} = B^n \times 0^q\,. \tag{11.28}$$

It is known that the Steiner polynomials $S_V(t)$ depends on the set V continuously: see Section 8 and footnote 24. Therefore,

$$\lim_{\varepsilon\to 0} S^{\mathbb{R}^{n+q}}_{E_{n,\,q,\,\varepsilon}}(t) = S^{\mathbb{R}^{n+q}}_{B^n\times 0^q}(t) \tag{11.29}$$

locally uniformly in $\mathbb{C}$. Hence, there exists an $\varepsilon(q,n),\ \varepsilon(q,n)>0$ such that the Steiner polynomial $S^{\mathbb{R}^{n+q}}_{E_{n,\,q,\,\varepsilon}}$ has roots located in the open right half plane. $\square$

The Weyl polynomials for the boundary surfaces of the adjoint convex sets. Keeping Definition 1.17 in mind, we now define the adjoint Weyl polynomials $W^{\,p}_{V\times 0^q}$.

Definition 11.10. Given a convex compact set V, $V\subset\mathbb{R}^n$, and a number q, $q = 0,\,1,\,2,\,3,\,\dots$, , the q-th adjoint Weyl polynomial $W^{\,1}_{\partial(V\times 0^q)}$ of the index 1 for the convex surface $\partial(V\times 0^q)$ is defined using the odd part of the q-th adjoint Steiner polynomial $S^{\mathbb{R}^{n+q}}_{V\times 0^q}$:

$$2tW^{\,1}_{\partial(V\times 0^q)}(t) \stackrel{\text{def}}{=} S^{\mathbb{R}^{n+q}}_{V\times 0^q}(t) - S^{\mathbb{R}^{n+q}}_{V\times 0^q}(-t)\,, \tag{11.30}$$

where $S^{\mathbb{R}^{n+q}}_{V^{(q)}}$ is the q-th adjoint Steiner polynomial which was introduced in Definition 11.1. More specifically [27] ,

$$W^{\,1}_{\partial(V\times 0^q)}(t) = \sum_{l\in\mathbb{Z}} s^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)t^{2l}\,. \tag{11.31}$$

Using (11.31), we can define the Weyl coefficients $w_{2l}(\partial(V\times 0^q))$ according to Definition 1.20:

$$w_{2l}(\partial(V\times 0^q)) = s^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)\frac{(2\pi)^l\omega_1}{\omega_{1+2l}}\,. \tag{11.32}$$

The Weyl polynomials $W^{p}_{\partial(V\times 0^q)}$ with higher p are then defined according to [28] Definition 1.24:

Definition 11.11.

$$W^{p}_{\partial(V\times 0^q)}(t) \stackrel{\text{def}}{=} \sum_{l\in\mathbb{Z}} w_{2l}(\partial(V\times 0^q))(2\pi)^{-l}\frac{\omega_{2l+p}}{\omega_p}t^{2l}\,. \tag{11.33}$$

Thus,

$$\omega_p t^p W^{p}_{\partial(V\times 0^q)}(t) = \sum_{l\in\mathbb{Z}} \frac{s^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)}{\omega_{2l+1}}\,\omega_1\omega_{2l+p}\,t^{2l+p}\,. \tag{11.34}$$

[27] According to the convention (11.22), $s^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q) = 0$ for $2l+1<0$ or $2l+1>n+q$.

[28] We remark that $\dfrac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)} = (2\pi)^{-l}\,\dfrac{\omega_{2l+p}}{\omega_p}$.

Let us clarify how the Weyl polynomials for the convex surfaces ∂V and $\partial(V\times 0^q)$ are related. Here, we also have to distinguish between even and odd cases for q.

Lemma 11.12. *Let V, $V\subset\mathbb{R}^n$, be a solid compact convex set, and let $p>0$, $q>0$ be integers. Then*

1. *For even q*

$$\omega_p t^p\cdot W^p_{\partial(V\times 0^q)}(t)=\omega_{p+q}t^{p+q}\cdot W^{p+q}_{\partial V}(t)\,; \tag{11.35}$$

2. *For odd q*

$$\omega_p t^p\cdot W^p_{\partial(V\times 0^q)}(t)=\omega_{p+q-1}t^{p+q-1}W^{p+q-1}_{\partial(V\times 0^1)}(t)\,. \tag{11.36}$$

Proof of Lemma 11.12.

1. Let q be even. The equality (11.21) with $k=2l+1-q$ takes the form

$$\frac{s^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)}{\omega_{2l+1}}=\frac{s^{\mathbb{R}^n}_{2l+1-q}(V)}{\omega_{2l+1-q}}.$$

From this and (11.34) it follows

$$\omega_p t^p W^p_{\partial(V\times 0^q)}(t)=\sum_{l\in\mathbb{Z}}\frac{s^{\mathbb{R}^n}_{2l+1-q}(V)}{\omega_{2l+1-q}}\,\omega_1\omega_{2l+p}\,t^{2l+p}\,.$$

Changing the summation variable: $l\to l+\frac{q}{2}$, we obtain

$$\omega_p t^p W^p_{\partial(V\times 0^q)}(t)=\sum_{l\in\mathbb{Z}}\frac{s^{\mathbb{R}^n}_{2l+1}(V)}{\omega_{2l+1}}\,\omega_1\omega_{2l+p+q}\,t^{2l+p+q}\,.$$

The expression on the right hand side of the latter equality has the same form as the expression on the right hand side of (11.34). One need only replace $V\times 0$ with V, p with $p+q$, and $n+q$ with n in (11.34) to see this. It follows that (11.35) holds.

2. Let q be odd. The equality implies the equality

$$\frac{s^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)}{\omega_{2l+1}}=\frac{s^{\mathbb{R}^{n+1}}_{2l+1-(q-1)}(V\times 0^1)}{\omega_{2l+1-(q-1)}}.$$

From this and (11.34) it follows that

$$\omega_p t^p W^p_{\partial(V\times 0^q)}(t)=\sum_{l\in\mathbb{Z}}\frac{s^{\mathbb{R}^{n+1}}_{2l+1-(q-1)}(V\times 0^1)}{\omega_{2l+1-(q-1)}}\,\omega_1\omega_{2l+p}\,t^{2l+p}\,.$$

Changing the summation variable: $l\to l+\frac{q-1}{2}$, we obtain

$$\omega_p t^p W^p_{\partial(V\times 0^q)}(t)=\sum_{l\in\mathbb{Z}}\frac{s^{\mathbb{R}^{n+1}}_{2l+1}(V\times 0^1)}{\omega_{2l+1}}\,\omega_1\omega_{2l+p+q-1}\,t^{2l+p+q-1}\,.$$

The expression on the right hand side of the latter equality has the same structure that the expression on the right hand side of (11.34). Again, one need only replace

$V \times 0^q$ with $V \times 0^1$, p with $p+q-1$, and $n+q$ with $n+1$. It follows that (11.36) holds. □

Lemma 11.12 tell us that when considering the boundary surfaces $\partial(V \times 0^q)$ of the q-th adjoint convex sets $V \times 0^q$, it is enough to restrict our considerations to the cases $q=0$ and $q=1$.

Proof of Statement 2 of Theorem 2.12. By Statement 2 of Theorem 6.8, the entire function $\mathcal{W}^{p+q-1}_{\partial(B^\infty \times 0)}$ has infinitely many non-real roots. (We assume that $p+q-1 \geq 5$.) If n is large enough, the Jensen polynomial $\mathcal{W}^{p+q-1}_{\partial(B^{n+1} \times 0)}(t) = \mathcal{J}_{2[n/2]}(\mathcal{W}^{p+q-1}_{\partial(B^\infty \times 0)}; t)$ also has non-real roots. According to (4.12b) and (4.13b), the Weyl polynomial $W^{p+q-1}_{\partial(B^n \times 0)}(t)$ has roots which do not belong to the imaginary axis. By Statement 2 of Lemma 11.12,

$$W^p_{\partial(B^n \times 0^q)} = \tfrac{\omega_{p+q-1}}{\omega_p} t^{q-1} W^{p+q-1}_{\partial(B^n \times 0)}\,.$$

Thus, the Weyl polynomial $W^p_{\partial(B^n \times 0^q)}$ has roots which do not belong to the imaginary axis. For fixed q, n and a positive ε, consider the ellipsoid $E_{n,\,q,\,\varepsilon}$ defined by (2.9). Since $E_{n,\,q,\,\varepsilon} \to B^n \times 0^q$ as $\varepsilon \to +0$, also $W^p_{E_{n,\,q,\,\varepsilon}} \to W^p_{\partial(B^n \times 0^q)}$ as $\varepsilon \to +0$. Hence, if ε is small enough: $0 < \varepsilon \leq \varepsilon(n,p,q)$, the polynomial $W_{E_{n,\,q,\,\varepsilon}}$ has roots which do not belong to the imaginary axis. □

12. The Steiner polynomial associated with the Cartesian product of convex sets.

Let V_1 and V_2 be compact convex sets,

$$V_1 \subset \mathbb{R}^{n_1}, \quad V_2 \subset \mathbb{R}^{n_2}.$$

Then the Cartesian product $V_1 \times V_2$ is a compact convex set embedded in the Cartesian product $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Since $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ can be naturally identified with $\mathbb{R}^{n_1+n_2}$, we can consider $V_1 \times V_2$ as being embedded in $\mathbb{R}^{n_1+n_2}$:

$$V_1 \times V_2 \subset \mathbb{R}^{n_1+n_2}.$$

The natural question arises:
How the Steiner polynomial $S^{\mathbb{R}^{n_1+n_2}}_{V_1 \times V_2}$ for the Cartesian product $V_1 \times V_2$ can be expressed in terms of the Steiner polynomials[29] $S^{\mathbb{R}^{n_1}}_{V_1}$ *and* $S^{\mathbb{R}^{n_2}}_{V_2}$ *for the Cartesian factors V_1 and V_2 ?*
To answer this question, we introduce a special multiplication operation in the set of polynomials, the so-called *M-product.*

[29] The Steiner polynomials S_{V_1}, S_{V_2}, $S_{V_1 \times V_2}$ are considered with respect to the ambient spaces $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, $\mathbb{R}^{n_1+n_2}$ respectively.

Definition 12.1. The M-product $t^k \circ t^l$ of two monomials t^k and t^l is defined as

$$t^k \circ t^l \overset{\text{def}}{=} \frac{\Gamma\big(\frac{k}{2}+1\big)\Gamma\big(\frac{l}{2}+1\big)}{\Gamma\big(\frac{k+l}{2}+1\big)}\, t^{k+l}\,, \quad k \geq 0,\, l \geq 0. \tag{12.1}$$

It is clear that

$$\text{a). } t^0 \circ t^k = t^k, \quad \text{b). } t^k \circ t^l = t^l \circ t^k, \quad \text{c). } (t^k \circ t^l) \circ t^m = t^k \circ (t^l \circ t^m). \tag{12.2}$$

The M-multiplication (12.1) of monomials can be extended to the multiplication of polynomials by linearity:

$$\text{For} \quad A(t) = \sum_{0 \leq k \leq n_1} a_k t^k, \quad B(t) = \sum_{0 \leq l \leq n_2} b_l t^l, \tag{12.3a}$$

$$(A \circ B)(t) = \sum_{\substack{0 \leq k \leq n_1,\\ 0 \leq l \leq n_2}} a_k b_l (t^k \circ t^l) = \sum_{0 \leq k \leq n_1, 0 \leq l \leq n_2} a_k b_l \frac{\Gamma\big(\frac{k}{2}+1\big)\Gamma\big(\frac{l}{2}+1\big)}{\Gamma\big(\frac{k+l}{2}+1\big)}\, t^{k+l}.$$

Finally, the M-product $A \circ B$ of the polynomials A and B is defined as

$$(A \circ B)(t) = \sum_{0 \leq r \leq n_1+n_2} \Bigg(\sum_{k \geq 0,\, l \geq 0, k+l=r} a_k\, b_l\, \frac{\Gamma\big(\frac{k}{2}+1\big)\Gamma\big(\frac{l}{2}+1\big)}{\Gamma\big(\frac{k+l}{2}+1\big)} \Bigg) t^r\,. \tag{12.3b}$$

From (12.2.b) and (12.2.c) it follows that

$$A \circ B = B \circ A, \quad (A \circ B) \circ C = A \circ (B \circ C)$$

for any polynomials A, B, C. In particular, the product $A \circ B \circ C$ of A, B, C is well defined. This product can be explicitly expressed in terms of the coefficients of its factors: if

$$A(t) = \sum_{0 \leq k \leq n_1} a_k t^k, \quad B(t) = \sum_{0 \leq l \leq n_2} b_l t^l, \quad C(t) = \sum_{0 \leq m \leq n_3} c_m t^m,$$

then

$$(A \circ B \circ C)(t) = \sum_{0 \leq r \leq n_1+n_2+n_3} \Bigg(\sum_{\substack{k \geq 0,\, l \geq 0,\, m \geq 0\\ k+l+m=r}} a_k\, b_l\, c_m \frac{\Gamma\big(\frac{k}{2}+1\big)\Gamma\big(\frac{l}{2}+1\big)\Gamma\big(\frac{m}{2}+1\big)}{\Gamma\big(\frac{k+l+m}{2}+1\big)} \Bigg) t^r\,.$$

It is clear that for every number λ and for any polynomials A and B,

$$(\lambda A) \circ B = \lambda (A \circ B).$$

Moreover, if

$$\mathbb{I}(t) \equiv 1, \quad \mathbb{T}(t) \equiv t,$$

then

$$\mathbb{I} \circ A = A.$$

Thus, *the polynomial* $\mathbb{I}$ *is the identity element for the* M*-product*
It is worth mentioning that

$$t^{(\circ\, k)} \stackrel{\text{def}}{=} \underbrace{t \circ t \circ \;\cdots\; \circ t}_{k} = \frac{(\sqrt{\pi}/2)^k}{\Gamma(\frac{k}{2}+1)} t^k. \tag{12.4}$$

Remark 12.2. M-multiplication by the polynomial $\mathbb{T}$ is related to the transformation (11.10):

$$\text{If} \qquad A(t) = \sum_{0 \le k \le n} a_k t^k, \tag{12.5a}$$

$$\text{then} \qquad (\underbrace{\mathbb{T} \circ \;\cdots\; \circ \mathbb{T}}_{p} \circ A)(t) = 2^{-p} t^p \sum_{0 \le k \le n} a_k \gamma_k^{(p)} t^k, \tag{12.5b}$$

where the 'multipliers' $\gamma_k^{(p)}$ are defined by (11.7).

Lemma 12.3. *The M-product* $A \circ B$ *of polynomials* A *and* B *admits the integral*[30] *representation*[31]

$$(A \circ B)(t) = A(0)B(t) + \int_0^t A\big((t^2 - \tau^2)^{1/2}\big)\, dB(\tau)\,, \tag{12.6a}$$

as well as

$$(A \circ B)(t) = A(t)B(0) + \int_0^t B\big((t^2 - \tau^2)^{1/2}\big)\, dA(\tau)\,. \tag{12.6b}$$

Proof. First of all, the expressions on the right hand sides of (12.6) are equal: Integrating by parts and replacing the variable $\tau \to (t^2 - \tau^2)^{1/2}$, we obtain

$$A(0)B(t) + \int_0^t A\big((t^2 - \tau^2)^{1/2}\big)\, dB(\tau) = A(t)B(0) + \int_0^t B\big((t^2 - \tau^2)^{1/2}\big)\, dA(\tau).$$

The expressions on the right hand sides of (12.6), which, at the first glance, are asymmetric with respect to A and B, are therefore actually symmetric. Let

$$A(t) = \sum_{0 \le k \le n_1} a_k t^k, \quad B(t) = \sum_{0 \le l \le n_2} b_l t^l$$

[30]The integrals on the right hand sides of (12.6) are Stieltjes integrals.
[31]At least, for $t > 0$.

be the expressions for the polynomials A and B in terms of their coefficients. Let us substitute these polynomials into the right hand side of (12.6a):

$$A(0)B(t) + \int\limits_0^t A\big((t^2-\tau^2)^{1/2}\big)\,dB(\tau) =$$

$$a_0 \sum_{0\le l\le n_2} b_l t^l + \int\limits_0^t \Big(\sum_{0\le k\le n_1} a_k (t^2-\tau^2)^{k/2}\Big)\cdot\Big(\sum_{1\le l\le n_2} l\, b_l \tau^{l-1}\Big)\,d\tau =$$

$$\sum_{0\le l\le n_2} a_0 b_l t^l + \sum_{\substack{0\le k\le n_1\\ 1\le l\le n_2}} a_k b_l \cdot l \int\limits_0^t (t^2-\tau^2)^{k/2}\tau^{l-1} d\tau\,.$$

Changing variable $\tau \to t\tau^{1/2}$, we get

$$l\int\limits_0^t (t^2-\tau^2)^{k/2}\tau^{l-1}d\tau = t^{k+l}(l/2)\int\limits_0^1 (1-\tau)^{k/2}\tau^{l/2-1}d\tau =$$

$$t^{k+l}(l/2)B\Big(\frac{k}{2}+1;\frac{l}{2}\Big)\,.$$

Now, according to Euler,

$$(l/2)B\Big(\frac{k}{2}+1;\frac{l}{2}\Big) = \frac{\Gamma\big(\frac{k}{2}+1\big)\frac{l}{2}\Gamma\big(\frac{l}{2}\big)}{\Gamma\big(\frac{k+l}{2}+1\big)} = \frac{\Gamma\big(\frac{k}{2}+1\big)\Gamma\big(\frac{l}{2}+1\big)}{\Gamma\big(\frac{k+l}{2}+1\big)}\,.$$

Thus, the right hand side of (12.6a) can be transformed into the right hand side of (12.3b). □

Theorem 12.4. *Given the compact convex sets V_1 and V_2, $V_1 \subset \mathbb{R}^{n_1}, V_2 \subset \mathbb{R}^{n_2}$, let $S_{V_1}^{\mathbb{R}^{n_1}}(t)$, $S_{V_2}^{\mathbb{R}^{n_2}}(t)$ be the Steiner polynomials for the sets V_1 and V_2. Then the Steiner polynomial $S_{V_1\times V_2}^{\mathbb{R}^{n_1+n_2}}$ of the Cartesian product $V_1\times V_2$ is equal to the M-product of the polynomials $S_{V_1}^{\mathbb{R}^{n_1}}$ and $S_{V_2}^{\mathbb{R}^{n_1 2}}$:*

$$S_{V_1\times V_2}^{\mathbb{R}^{n_1+n_2}} = S_{V_1}^{\mathbb{R}^{n_1}} \circ S_{V_2}^{\mathbb{R}^{n_2}}\,. \tag{12.7}$$

A sketch of the proof for this theorem can be found in [Had, Chapter VI, Section 6.1.9]. A detailed proof is presented below.

Remark 12.5. Let S be the subset of $\mathbb{R}^1$ containing only the origin point, i.e., $S=\{t\in\mathbb{R}^1 : t=0\}$. Then $S_S(t)=2t$, that is,

$$S_S(t) = 2\,\mathbb{T}(t). \tag{12.8}$$

Let V be a compact convex set embedded into $\mathbb{R}^n$. The Cartesian product $V \times \underbrace{S \times \cdots \times S}_{p}$ can be identified with the convex set $V \times 0^p,\ V \times 0^p \subset \mathbb{R}^{n+p}$. Thus,

$$S_{V \times \underbrace{S \times \cdots \times S}_{p}}(t) = S^{\mathbb{R}^{n+p}}_{V \times 0^p}(t)\,,$$

or

$$2^p(\underbrace{\mathbb{T} \circ \cdots \circ \mathbb{T}}_{p}) \circ S^{\mathbb{R}^n}_{V} = S^{\mathbb{R}^{n+p}}_{V \times 0^p}. \tag{12.9}$$

In view of (12.5) and (12.8), the equality (12.9) is another form of the equality (11.6).

Proof of Theorem 12.4. Let

$$V = V_1 \times V_2\,.$$

According to the identification $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, we present a point $x \in \mathbb{R}^{n_1+n_2}$ as a pair $x = (x_1,\, x_2)$, where $x_1 \in \mathbb{R}^{n_1},\ x_2 \in \mathbb{R}^{n_2}$. It is clear that

$$\operatorname{dist}^2_{\mathbb{R}^{n_1+n_2}}(x,\, V) = \operatorname{dist}^2_{\mathbb{R}^{n_1}}(x_1,\, V_1) + \operatorname{dist}^2_{\mathbb{R}^{n_2}}(x_2,\, V_2). \tag{12.10}$$

For $\tau > 0$, $\tau' > 0$, $\tau'' > 0$, let $V(\tau), V_1(\tau')$ and $V_2(\tau'')$ be the τ-neighborhood of V with respect to $\mathbb{R}^{n_1+n_2}$, the τ'-neighborhood of V_1 w. r. t. $\mathbb{R}^{n_1}$ and τ''-neighborhood of V_2 w. r. t. $\mathbb{R}^{n_2}$ respectively:

$$\begin{gathered} V(\tau) = V + \tau B^{n_1+n_2}, \quad V_1(\tau') = V_1 + \tau' B^{n_1}, \quad V_2(\tau'') = V_2 + \tau'' B^{n_2}; \\ V,\, B^{n_1+n_2} \subset \mathbb{R}^{n_1+n_2}; \quad V_1,\, B^{n_1} \subset \mathbb{R}^{n_1}; \quad V_2,\, B^{n_2} \subset \mathbb{R}^{n_2}. \end{gathered}$$

Here $B^{n_1+n_2}$, B^{n_1}, B^{n_2} denote the Euclidean balls of the radius one in $\mathbb{R}^{n_1+n_2}$, $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, respectively.

Given a number t, $t > 0$, consider the t-neighborhood $V(t)$ of $V = V_1 \times V_2$, and let

$$0 = \tau_0 < \tau_1 < \ \ldots\ < \tau_{N-1} < \tau_N = t$$

be a subdivision of the interval $[0,\, t]$. From (12.10) it follows that

$$\begin{aligned} &\big(V_1(0) \times V_2(t)\big) \cup \Big(\bigcup_{1 \le k \le N} \big(V_1(\tau_k) \setminus V_1(\tau_{k-1})\big) \times V_2\big((t^2 - \tau_k^2)^{1/2}\big)\Big) \\ &\qquad\qquad \subseteq V(t) \subseteq \\ &\big(V_1(0) \times V_2(t)\big) \cup \Big(\bigcup_{1 \le k \le N} \big(V_1(\tau_k) \setminus V_1(\tau_{k-1})\big) \times V_2\left((t^2 - \tau_{k-1}^2)^{1/2}\right)\Big)\,. \end{aligned} \tag{12.11}$$

Since $V_1(\tau_k) \supseteq V_1(\tau_{k-1})$,

$$\operatorname{Vol}_{n_1}\big(V_1(\tau_k) \setminus V_1(\tau_{k-1}\big) = \operatorname{Vol}_{n_1}\big(V_1(\tau_k)\big) - \operatorname{Vol}_{n_1}\big(V_1(\tau_{k-1})\big),$$

thus

$$\mathrm{Vol}_{n_1+n_2}\Big(\big(V_1(\tau_k)\setminus V_1(\tau_{k-1})\big)\times V_2\left((t^2-\tau_l^2)^{1/2}\right)\Big)=$$
$$\Big(\mathrm{Vol}_{n_1}\big(V_1(\tau_k)\big)-\mathrm{Vol}_{n_1}\big(V_1(\tau_{k-1})\big)\Big)\cdot\mathrm{Vol}_{n_2}\Big(V_2\big((t^2-\tau_l^2)^{1/2}\big)\Big),$$
$$l=k-1 \text{ or } l=k.$$

Moreover,

$$\mathrm{Vol}_{n_1+n_2}\Big(V_1(0)\times V_2(t)\Big)=\mathrm{Vol}_{n_1}\big(V_1(0)\big)\cdot\mathrm{Vol}_{n_2}\big(V_2(t)\big)\,.$$

In the notation of Steiner polynomials, the latter equalities take the form

$$\mathrm{Vol}_{n_1+n_2}\Big(\big(V_1(\tau_k)\setminus V_1(\tau_{k-1})\big)\times V_2\left((t^2-\tau_l^2)^{1/2}\right)\Big)=$$
$$\Big(S_{V_1}(\tau_k)-S_{V_1}(\tau_{k-1})\Big)\cdot S_{V_2}\big((t^2-\tau_l^2)^{1/2}\big),\quad l=k-1 \text{ or } l=k, \tag{12.12a}$$
$$\mathrm{Vol}_{n_1+n_2}\Big(V_1(0)\times V_2(t)\Big)=S_{V_1}(0)\cdot S_{V_2}(t)\,, \tag{12.12b}$$

and

$$\mathrm{Vol}_{n_1+n_2}\big(V(t)\big)=S_V(t). \tag{12.12c}$$

Since the sets $V_1(\tau_k)\setminus V_1(\tau_{k-1})$ do not intersect for pairwise different k, and none of these sets intersects with the set $V_1(0)$, it follows from (12.11) and (12.12) that

$$S_{V_1}(0)\cdot S_{V_2}(t)+\sum_{1\le k\le N}\Big(S_{V_1}(\tau_k)-S_{V_1}(\tau_{k-1})\Big)\cdot S_{V_2}\big((t^2-\tau_k^2)^{1/2}\big)$$
$$\le S_V(t)\le \tag{12.13}$$
$$S_{V_1}(0)\cdot S_{V_2}(t)+\sum_{1\le k\le N}\Big(S_{V_1}(\tau_k)-S_{V_1}(\tau_{k-1})\Big)\cdot S_{V_2}\big((t^2-\tau_{k-1}^2)^{1/2}\big)\,.$$

Passing to the limit as $\max(\tau_k-\tau_{k-1})\to 0$ in the latter inequality, we express the Steiner polynomial $S_V(t)$ as the Stieltjes integral

$$S_V(t)=S_{V_1}(0)\cdot S_{V_2}(t)+\int\limits_0^t S_{V_2}\big((t^2-\tau^2)^{1/2}\big)\,dS_{V_1}(\tau)\,. \tag{12.14}$$

According to Lemma 12.3, the expression on the right hand side of (12.14) is equal to $\big(S_{V_1}\circ S_{V_2}\big)(t)$. □

13. Properties of entire functions generating the Steiner and Weyl polynomials for the degenerate convex sets $B^{n+1}\times 0^q$.

In this section we investigate the locations of roots for entire functions generating the Steiner and Weyl polynomials related to the ‘degenerate’ convex sets $B^{n+1}\times 0^q$. Entire functions of this kind are

- The entire functions $\mathcal{M}_{B^\infty\times 0^q}$ which appear in (11.26), in particular for $q=1$ in (4.7b) .
- The entire function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$, $1\le p<\infty$, which appears in (4.11c) .
- The entire function $\mathcal{W}^\infty_{\partial(B^\infty\times 0)}$ which appears in (4.11d) .

The functions $\mathcal{M}_{B^\infty\times 0^q}$, $\mathcal{W}^p_{\partial(B^\infty\times 0)}$, $1\le p<\infty$ can not be determined explicitly (except for some special values of the parameters p or q), but they can be calculated asymptotically.

The above-mentioned functions admit integrable representations:

$$\mathcal{M}_{B^\infty\times 0^q}(t)=q\int\limits_0^1(1-\xi^2)^{\frac{q}{2}-1}\xi\, e^{\xi t}\,d\xi\,; \tag{13.1}$$

$$\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)=p\int\limits_0^1(1-\xi^2)^{\frac{p}{2}-1}\xi\cos t\xi\,d\xi\,; \tag{13.2}$$

These integral representations can be used for the asymptotic calculation of the functions $\mathcal{M}_{B^\infty\times 0^q}$, $\mathcal{W}^p_{\partial(B^\infty\times 0)}$.

Another way to calculate the functions (13.1), (13.2) asymptotically is to use the structure of their respective Taylor series. The Taylor coefficients of each of these functions are ratios of factorials. These functions belong to the so-called class of *Fox-Wright functions*, [CrCs4].

The Fox-Wright class of functions is defined as a class of functions of the form

$${}_p\Psi_q\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & & \alpha_p\\ \beta_1 & \beta_2 & & \beta_p\end{smallmatrix};\begin{smallmatrix}\rho_1 & \rho_2 & & \rho_q\\ \sigma_1 & \sigma_2 & & \sigma_q\end{smallmatrix};z\right\}\overset{\mathrm{def}}{=}\sum_{0\le k<\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j\,k+\beta_j)}{\prod_{j=1}^{q}\Gamma(\rho_j\,k+\sigma_j)}\cdot\frac{z^k}{k!}\,. \tag{13.3}$$

Comparing (13.3) with the Taylor expansions (11.26), (4.11c), (4.11d), we see that

$$\mathcal{M}_{B^\infty\times 0^q}(t)=\Gamma\left(\frac{q}{2}+1\right)\cdot{}_1\Psi_1\left\{\begin{smallmatrix}\frac12\\ 1\end{smallmatrix};\begin{smallmatrix}\frac12\\ 1+\frac{q}{2}\end{smallmatrix};t\right\},\quad 1\le q<\infty\,, \tag{13.4}$$

$$\mathcal{W}^p_{\partial(B^\infty\times 0^1)}(t)=\Gamma(\tfrac12)\cdot\Gamma(\tfrac{p}{2}+1)\cdot{}_1\Psi_2\left\{\begin{smallmatrix}1\\ 1\end{smallmatrix};\begin{smallmatrix}1 & 1\\ \frac12 & \frac{p}{2}+1\end{smallmatrix};-\frac{t^2}{4}\right\},\quad 1\le p<\infty\,, \tag{13.5}$$

E. Barnes, [Bar], G.N. Watson, [Wat], G. Fox, citeFox, as well as E.M. Wright, [Wr1], [Wr2], have all studied the asymptotic behavior of the function ${}_p\Psi_q(z)$.

Analysis of the function $\mathcal{M}_{B^\infty\times 0^q}(t)$: We would like to determine for which q this function belongs to the Hurwitz class $\mathcal{H}$. According to (13.4), we may reduce the question to the proportional function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac12\\ 1\end{smallmatrix};\begin{smallmatrix}\frac12\\ 1+\frac{q}{2}\end{smallmatrix};z\right\}$. From the Taylor expansion it is clear that this function is an entire function of exponential type. Since the Taylor coefficients of the function are positive, its defect [32] is non-negative. *The function $\mathcal{M}_{B^\infty\times 0^q}(t)$ is therefore in the Hurwitz class $\mathcal{H}$ only if all roots of*

[32] We recall that the defect of an entire function H of exponential type is defined by (5.1).

the function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ *are situated in the open left half plane.* To investigate the locations of roots for the latter function, we use the following asymptotic approximation, which can be derived from results in [Wr1], [Wr2]:
For any $\varepsilon:\ 0<\varepsilon<\dfrac{\pi}{2}$,

$$
\begin{aligned}
&{}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}=\\
&\qquad=\begin{cases}2^{\frac{q}{2}}z^{-\frac{q}{2}}e^{z}\left(1+r_1(z)\right), & |\arg z|\leq\frac{\pi}{2}-\varepsilon\,;\\ \frac{2}{\Gamma(\frac{q}{2})}z^{-2}+r_2(z), & |\arg z-\pi|\leq\frac{\pi}{2}-\varepsilon\,;\\ 2^{\frac{q}{2}}z^{-\frac{q}{2}}e^{z}+\frac{2}{\Gamma(\frac{q}{2})}z^{-2}+r_3(z)\,, & |\arg z\mp\frac{\pi}{2}|\leq\varepsilon\,.\end{cases}
\end{aligned}
\tag{13.6}
$$

The remainders can be estimated as follows:

$$
\begin{aligned}
&|r_1(z)|\leq C_1(\varepsilon)|z|^{-1},|\arg z|\leq\frac{\pi}{2}-\varepsilon\,;\\
&\qquad|r_2(z)|\leq C_2(\varepsilon)|z|^{-3},|\arg z-\pi|\leq\frac{\pi}{2}-\varepsilon\,;\\
&\qquad\qquad|r_3(z)|\leq C_3\left(|z|^{-3}+|e^{z}||z|^{-(\frac{q}{2}+1)}\right)\,,\quad|\arg z\mp\frac{\pi}{2}|\leq\varepsilon\,,
\end{aligned}
\tag{13.7}
$$

where the values $C_1(\varepsilon)<+\infty,\ C_2(\varepsilon)<+\infty,\ C_3(\varepsilon)<+\infty$ do not depend on z.

From (13.6), (13.7) it follows that for any $\varepsilon>0$, not more than finitely many roots of the function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ lie outside of the angular domain $\{z:\ |\arg z\mp\frac{\pi}{2}|\leq\varepsilon$. These roots are asymptotically close to the roots of the approximating function $f_q(z)$:

$$
f_q(z)=2^{\frac{q}{2}}z^{-\frac{q}{2}}\left(e^{z}+\frac{2^{1-\frac{q}{2}}}{\Gamma(\frac{q}{2})}z^{\frac{q}{2}-2}\right).
$$

One should distinguish several cases when examining the roots of the approximating function $f_q(z)$.
$\boldsymbol{q=4}$. In this case, the equation $f_q(z)=0$ becomes $e^z+\frac{1}{2}=0$, so, the roots of the approximating function can be found explicitly. These root form an arithmetical progression located on the straight line $\{z=x+iy:\ x=-\ln 2,\ -\infty<y<\infty\}$. From this and (13.6)-(13.7) it follows that the roots of the the analyzed function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ are asymptotically close to the above-mentioned straight line. Thus, *for $q=4$ all but finitely many roots of the function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ are located in the open left half plane.* In fact, for $q=4$ ***all*** roots of this function are located in the open left half plane. To establish this, further considerations are needed. We follow up on this later.
$\boldsymbol{q\neq4}$. In this case, the equation $f_q(z)=0$ becomes the equation

$$
e^{z}+c_q\,z^{\frac{q}{2}-2}=0\,,\quad c_q=\frac{2^{1-\frac{q}{2}}}{\Gamma(\frac{q}{2})}\,,
$$

where the exponent $\frac{q}{2}-2$ is non-zero. The latter equation has infinitely many roots which have no finite accumulation points and which are asymptotically close to the 'logarithmic parabola'

$$x = (\tfrac{q}{2}-1)\ln(|y|+1) + ln|c|, \ -\infty < y < \infty\,, \quad (z = x+iy). \tag{13.8}$$

From this and from (13.6)-(13.7) it follows that the roots of the function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ are asymptotically close to the logarithmic parabola (13.8). Now we should distinguish the cases $q<4$ and $q>4$.

$\boldsymbol{q<4}$. In this case, the part of the logarithmic parabola (13.8) lying outside some compact set is located inside the left half plane. Since the roots of the analyzed function are asymptotically close to this parabola, all but finitely many roots are located in the left half plane.

$\boldsymbol{q>4}$. In this case, the part of the logarithmic parabola (13.8) lying outside some compact set is located inside the right half plane. Therefore all but finitely many roots of the function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ are located in the right half plane.

We formulate this result as

Lemma 13.1. *If $q>4$, then the entire function $\mathcal{M}_{B^\infty\times 0^q}$ has infinitely many roots within the right half plane. In particular, this function does not belong to the Hurwitz class $\mathcal{H}$.*

Claim 2 of Lemma 11.9 is a consequence of Lemma 13.1.

Lemma 13.2. *For $q: 0\le q\le 2$, the function $\mathcal{M}_{B^\infty\times 0^q}$ belongs to the Hurwitz class $\mathcal{H}$.*

Proof. For $q=0$, the assertion is clear: the function in question is equal to e^z. For $q>0$, Lemma 13.2 is a consequence of Lemma 13.3. □

To investigate the case $q>0$, we use the integral representation

$${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\} = \frac{q}{\Gamma(\frac{q}{2}+1)}\,I_q(z)\,, \tag{13.9}$$

where

$$I_q(z) = \int\limits_0^1 (1-\xi^2)^{\frac{q}{2}-1}\xi\, e^{\xi t}\,d\xi\,. \tag{13.10}$$

The defect of the entire function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ is non-negative, so it is enough to prove that this function has no roots in the closed right half plane. The function $I_q(z)$ is of the form

$$I_q(z) = \int\limits_0^1 \varphi_q(\xi)e^{\xi z}d\xi\,, \tag{13.11}$$

where

$$\varphi_q(\xi) = (1-\xi^2)^{\frac{q}{2}-1}\xi, \quad 0\le\xi\le 1\,. \tag{13.12}$$

The crucial case is:
For $q: \ 0 \le q \le 2$, *the function* $\varphi_q(\xi)$ *is positive and strictly increasing on the interval* $(0,1)$.

Lemma 13.3. [Pólya] *If* $\varphi(\xi)$ *is a non-negative increasing function on the interval* $[0,1]$, *then the entire function*

$$I(z) = \int\limits_0^1 \varphi(\xi)e^{\xi z}d\xi \tag{13.13}$$

has no zeros in the closed right half plane.

This lemma is a continuous generalization of a theorem by S. Kakeya. Proof of this Lemma and the reference to the paper of S. Kakeya could be found in [Po1, § 1]. We give another proof, based on an approach suggested by [OsPe, Lemma 4].

Proof of Lemma 13.3. Let $z = x + iy$. Since $I(x) > 0$ for $x \ge 0$, $I(x)$ has no zeros for $0 \le x < \infty$. Let us show that $I(z)$ has no zeros for $0 \le x < \infty$, $y \ne 0$. It is enough to consider the case $y > 0$. We prove that $\operatorname{Im}(e^{-z}I(z)) < 0$ for $z = x + iy, \ x \ge 0, y > 0$, thus $I(z) \ne 0$ for $z = x + iy, \ x \ge 0, y > 0$. To prove this, we use the integral representation

$$e^{-z}I(z) = \int\limits_0^\infty \psi(\xi)e^{-i\xi y}d\xi\,, \tag{13.14}$$

where

$$\psi(\xi) = \begin{cases} \varphi(1-\xi)e^{-x\xi}, & 0 \le \xi \le 1\,, \\ 0, & 1 < \xi < \infty\,. \end{cases} \tag{13.15}$$

In particular,

$$-\left(\operatorname{Im} e^{-z}I(z)\right) = \int\limits_0^\infty \psi(\xi)\sin\xi y\, d\xi\,, \quad z = x+iy, \quad x \ge 0,\, y > 0\,, \tag{13.16}$$

where the function $\psi(\xi)$ is non-negative and decreasing on $[0,\infty)$, strictly decreasing on some non-empty open interval, and $\psi(\infty) = 0$. Further,

$$\int\limits_0^\infty \psi(\xi)\sin\xi y\, d\xi = \sum_{k=0}^\infty \int\limits_{\frac{k\pi}{y}}^{\frac{(k+1)\pi}{y}} \psi(\xi)\sin\xi y\, d\xi =$$

$$\int\limits_0^{\frac{\pi}{y}} \left(\sum_{k=0}^\infty (-1)^k\psi(\xi + \tfrac{k\pi}{y})\right)\sin\xi y\, d\xi > 0\ : \tag{13.17}$$

The series in the latter integral is a Leibnitz type series. Thus the sum of this series is non-negative on the interval of integration and is strictly positive on some subinterval. □

Lemma 13.4. *For $q = 4$, the function $\mathcal{M}_{B^\infty \times 0^q}$ belongs to the Hurwitz class $\mathcal{H}$.*

Proof. For $q = 4$, the integral in (13.10) can be calculated explicitly:

$$I_4(z) = \frac{(2z^2 - 6z + 6)e^z + (z^2 - 6)}{z^4} \,. \tag{13.18}$$

Our goal is to prove that the function $\frac{(2z^2-6z+6)e^z+(z^2-6)}{z^4}$ has no roots in the closed right half plane. Instead of considering of this function, we first focus on the function

$$f(z) = (2z^2 - 6z + 6) + (z^2 - 6)e^{-z} \,. \tag{13.19}$$

We will now prove that $f(z)$ has no roots in the closed right half plane, other than the root at the point $z = 0$ of multiplicity four. The function f is of the form

$$f(z) = g(z) + h(z) \,, \text{ where } \; g(z) = (2z^2 - 6z + 6), \; h(z) = (z^2 - 6)e^{-z} \,. \tag{13.20}$$

In the right half plane the function h is subordinate to the function g in the following sense. For $R > 0$, let us consider the contour Γ_R, which consists of the interval I_R of the imaginary axis and the semicircle C_R located in the right half plane:

$$\Gamma_R = I_R \cup C_R \,, \text{ where } I_R = [-iR, iR], \; C_R = \{z : \, |z| = R, \, \mathrm{Re}\, z \geq 0\} \,. \tag{13.21}$$

It is clear that $|g(z)| \geq 1.75\,|z|^2$, $|h(z)| \leq 1.25\,|z|^2$ if $z \in C_R$ and R is large enough. In particular, $|g(z)| > |h(z)|$ if $z \in C_R$ and R is large enough. On the imaginary axis, we have that

$$|g(iy)|^2 = 36 + 12y^2 + 4y^4, \quad |h(iy)|^2 = 36 + 12y^2 + y^4 \,, \; -\infty < y < \infty, \tag{13.22}$$

In particular, $|g(z)| \geq |h(z)|$ for $z \in I_R$, and the inequality is strict for $z \neq 0$. Thus,

$$|g(z)| \geq |h(z)| \quad \text{for} \quad z \in \Gamma_R \quad \text{and } R \text{ large enough.} \tag{13.23}$$

For $0 < \varepsilon < 1$, consider the function

$$f_\varepsilon(z) = g(z) + (1 - \varepsilon)h(z) \,. \tag{13.24}$$

The polynomial g has two simple roots: $z_{1,2} = \frac{3 \pm i\sqrt{3}}{2}$. They are located in the open right half plane. In view of (13.24) and Rouche's theorem, for $\varepsilon > 0$ the function $f_\varepsilon(z)$ has precisely two roots $z_1(\varepsilon)$, $z_2(\varepsilon)$ in the open right half plane. For ε positive and very small, the roots $z_1(\varepsilon)$, $z_2(\varepsilon)$ are located very close to the boundary point $z = 0$. This can be shown by the asymptotic calculation. Since $f_\varepsilon(z) = 6\varepsilon + \frac{1-\varepsilon}{4} z^4 + o(|z|^4)$ as $z \to 0$, the equation $f_\varepsilon(z) = 0$ has the roots $z_1(\varepsilon)$, $z_2(\varepsilon)$, for which

$$z_{1,2}(\varepsilon) = \varepsilon^{\frac{1}{4}} 24^{\frac{1}{4}} e^{\pm \frac{\pi}{4}} (1 + o(\varepsilon)) \;\text{ as } \varepsilon \to +0 \,.$$

Since the function f_ε has only two roots in the open right half plane for $\varepsilon > 0$, $z_1(\varepsilon)$, $z_2(\varepsilon)$ are the only roots located in this half plane. Since $f(z) = \lim_{\varepsilon \to +0} f_\varepsilon(z)$, the function $f(z)$ has no roots in the right upper half plane. (We apply Hurwitz's

Theorem.) From (13.22) it follows that the function f does not vanish on the imaginary axis except at $z=0$. At this point the function f has a root of multiplicity four. Thus, for $q=4$ the function $I_q(z)$ is in the Hurwitz class. $\square$

Claim 1 of Lemma 11.9 is a consequence of Lemma 13.2 and Lemma 13.4.

Remark 13.5. From (13.9) and (13.18) we obtain:

$$ {}_1\Psi_1\left\{\tfrac{1}{2}_{1};\ \tfrac{1}{2}_{1+\frac{q}{2}};z\right\} = 2\,\frac{(2z^2-6z+6)e^z+(z^2-6)}{z^4} \quad \text{for} \quad q=4\,. \tag{13.25}$$

This expression agrees with the asymptotic relation (13.6).

Analysis of the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$: We may calculate the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ asymptotically expressing it in terms of the appropriate Fox-Wright function, (13.5), and then make use of the asymptotic expansion of this Fox-Wright function. However, we derive the asymptotic approximation of the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ from the asymptotic approximation of the function $\mathcal{M}_{B^\infty\times 0^p}(t)$. From (13.1) and (13.2) it follows that

$$\mathcal{W}^p_{\partial(B^\infty\times 0)}(t) = \frac{1}{2}\big(\mathcal{M}_{B^\infty\times 0^p}(it)+\mathcal{M}_{B^\infty\times 0^p}(-it)\big)\,. \tag{13.26}$$

Comparing (13.26) with (13.6), we see that

$$\mathcal{W}^p_{\partial(B^\infty\times 0)}(t) = \\
= \begin{cases} 2^{\frac{p}{2}}\ t^{-\frac{p}{2}}\ \cos\left(t-\frac{\pi p}{4}\right)-\frac{2}{\Gamma(\frac{p}{2})}t^{-2}+r_1(t), & |\arg t|\le\varepsilon\,,\\ 2^{\frac{q}{2}}(te^{-i\pi})^{-\frac{p}{2}}\cos\left(t+\frac{\pi p}{4}\right)-\frac{2}{\Gamma(\frac{p}{2})}t^{-2}+r_2(t), & |\arg t-\pi|\le\varepsilon\,,\\ 2^{\frac{p}{2}}(t\,e^{\mp\frac{i\pi}{2}})^{-\frac{p}{2}}\,e^{\mp it}(1+r_3(t))\,, & |\arg t\pm\frac{\pi}{2}|\le\frac{\pi}{2}-\varepsilon\,, \end{cases} \tag{13.27}$$

where the remainders $r_1(t)$, $r_2(t)$, $r_3(t)$ can be estimated as follows

$$|r_1(t)|\le C_1(\varepsilon)\left(|t|^{-(1+\frac{p}{2})}e^{|\operatorname{Im} t|}+|t|^{-3}\right)\,, \qquad |\arg t|\le\varepsilon\,, \tag{13.28a}$$

$$|r_2(t)|\le C_2(\varepsilon)\left(|t|^{-(1+\frac{p}{2})}e^{|\operatorname{Im} t|}+|t|^{-3}\right)\,, \quad |\arg t-\pi|\le\varepsilon\,, \tag{13.28b}$$

$$|r_3(t)|\le C_3(\varepsilon)\,|t|^{-1}\,, \qquad |\arg t\mp\frac{\pi}{2}|\le\frac{\pi}{2}-\varepsilon\,, \tag{13.28c}$$

and $C_1(\varepsilon)<\infty$, $C_2(\varepsilon)<\infty$, $C_3(\varepsilon)<\infty$ for every $\varepsilon:\ 0<\varepsilon<\frac{\pi}{2}$. Moreover, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ is an even function of t and takes real values for real t.

From (13.27), (13.28) it follows that for every $\varepsilon>0$ the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ can not have more that finitely many roots within the angular domains $\{t:\ |\arg(t\mp\frac{\pi}{2}|\le\frac{\pi}{2}-\varepsilon\}$. Within the angular domain $\{t:\ |\arg t|\le\varepsilon\}$, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ has infinitely many roots, and these roots are asymptotically close to the roots of the approximating function

$$f_p(t)=2^{\frac{p}{2}}t^{-\frac{p}{2}}\cos\left(t-\tfrac{\pi p}{4}\right)-\frac{2}{\Gamma(\frac{p}{2})}t^{-2}\,,\quad |\arg t|\le\varepsilon\,.$$

(Since the function $\mathcal{W}^{p}_{\partial(B^\infty\times 0)}(t)$ is even, there is no need to study its behavior within the angular domain $\{t:\,|\arg t-\pi|\le\varepsilon\}$.) The behavior of thr roots for the approximating equation $f_p(t)=0$, that is, the equation

$$\cos\,(t-\tfrac{\pi p}{4})-\frac{2^{1-\frac{p}{2}}}{\Gamma(\frac{p}{2})}\,t^{\frac{p}{2}-2}=0\,,\quad |\arg t|\le\varepsilon\,, \tag{13.29}$$

depends on p.

If $0<p<4$, then all but finitely many roots of the equation (13.29) are real and simple, and these roots are asymptotically close to the roots of the equation $\cos\,(t-\frac{\pi p}{4})=0$.
If $p=4$, then all but finitely many roots of the equation (13.29) are real and simple and these roots are asymptotically close to the roots of the equation $\cos\,(t-\pi)-\frac{1}{2}=0$.
If $p>4$, then all but finitely many roots of the equation (13.29) are non-real and simple, located symmetrically with respect to the real axis, and are asymptotically close to the 'logarithmic parabola'

$$|y|=(\tfrac{p}{2}-1)\ln(|x|+1)+\ln c_p\,,\quad c_p=\frac{2^{1-\frac{p}{2}}}{\Gamma(\frac{p}{2})}\,,\quad 0\le x<\infty\,.$$

Thus we prove the following

Lemma 13.6. *For each $p:\ 0\le p<\infty$, the function $\mathcal{W}^{p}_{\partial(B^\infty\times 0)}(t)$ has infinitely many roots. All but finitely many these roots are simple. They lie symmetric with respect to the point $z=0$.*

1. *If $0\le p\le 4$, then all but finitely many these roots are real;*
2. *If $4<p$, then all but finitely many these roots are non-real. In particular, the function $\mathcal{W}^{p}_{\partial(B^\infty\times 0)}(t)$ does not belong to the Laguerre-Pólya class $\mathcal{L}$-$\mathcal{P}$.*

Lemma 13.7. *For $0<p\le 2$, as well as for $p=4$, the function $\mathcal{W}^{p}_{\partial(B^\infty\times 0)}$ belongs to the Laguerre-Pólya class.*

Proof. The equality (13.26) will serve as a starting point for our considerations. If the function $\mathcal{M}_{B^\infty\times 0^p}(t)$ is in the Hurwitz class $\mathcal{H}$, then the function

$$\Omega(t)=\mathcal{M}_{B^\infty\times 0^p}(it) \tag{13.30}$$

is in the class $\mathcal{P}$ (in the sense of [Lev1]).

Definition 13.8. [Lev1, Chapter VII, Section 4]. An entire function $\Omega(t)$ of exponential type belongs to the class $\mathcal{P}$ if:

1. $\Omega(t)$ has no roots in the closed lower half plane $\{t:\,\operatorname{Im}t\le 0\}$.
2. The defect d_Ω of the function Ω is non-negative, where

$$2d_\Omega=\varlimsup_{r\to+\infty}\frac{\ln|\Omega(-ir)|}{r}-\varlimsup_{r\to+\infty}\frac{\ln|\Omega(ir)|}{r}\,.$$

In the book [Lev1] of B.Ya.Levin, the following version of the Hermite-Biehler Theorem is proved:

Theorem. [Lev1, Chapter VII, Section 4, Theorem 7] *If an entire function $\Omega(t)$ is in class $\mathcal{P}$, then its real and imaginary parts, ${}^{\mathcal{R}}\Omega(t)$ and ${}^{\mathcal{I}}\Omega(t)$ respectively:*

$$ {}^{\mathcal{R}}\Omega(t) = \frac{\Omega(t) + \overline{\Omega(\overline{t})}}{2}, \quad {}^{\mathcal{I}}\Omega(t) = \frac{\Omega(t) - \overline{\Omega(\overline{t})}}{2i}, $$

possess the following properties:

1. *The roots of each of the functions ${}^{\mathcal{R}}\Omega(t)$ and ${}^{\mathcal{I}}\Omega(t)$ are real and simple;*
2. *The zero loci of the functions ${}^{\mathcal{R}}\Omega(t)$ and ${}^{\mathcal{I}}\Omega(t)$ interlace.*

Let us apply this theorem to the function $\Omega(t)$ defined by (13.30):

$$ \Omega(t) = \mathcal{M}_{B^\infty \times 0^p}(it) \,. $$

Taking into account that the function $\mathcal{M}_{B^\infty \times 0^p}$ is real:
$\mathcal{M}_{B^n \times 0^p}(t) \equiv \overline{\mathcal{M}_{B^\infty \times 0^p}(\overline{t})}$, or equivalently $\mathcal{M}_{B^n \times 0^p}(-it) \equiv \overline{\mathcal{M}_{B^\infty \times 0^p}(i\overline{t})}$. Hence,

$$ {}^{\mathcal{R}}\Omega(t) = \frac{1}{2}\big(\mathcal{M}_{B^n \times 0^p}(it) + \mathcal{M}_{B^\infty \times 0^p}(-it)\big)\,, $$

or because of (13.26):

$$ {}^{\mathcal{R}}\Omega(t) = \mathcal{W}^{p}_{\partial(B^\infty \times 0)}(t)\,. $$

Thus, the following result holds:

Lemma 13.9. *If the function $\mathcal{M}_{B^n \times 0^p}(t)$ belongs to the Hurwitz class $\mathcal{H}$, then the function $\mathcal{W}^{p}_{\partial(B^\infty \times 0)}(t)$ belongs to the Laguerre-Pólya class $\mathcal{L}$-$\mathcal{P}$.*

Combining Lemma 13.9 with Lemmas 13.2 and 13.4, we obtain Lemma 13.7.

□

Remark 13.10. The real part ${}^{\mathcal{R}}\Omega(t)$ for $\Omega(t)$ from (13.30), has infinitely many non-real roots if $p > 4$. Nevertheless, all roots of the imaginary part ${}^{\mathcal{I}}\Omega(t)$ are real for every $p \geq 0$.

Indeed, according to (13.1) and (13.30),

$$ {}^{\mathcal{I}}\Omega(t) = p\int\limits_0^1 (1-\xi^2)^{\frac{p}{2}-1}\xi \,\sin\xi t\, d\xi\,. \tag{13.31} $$

According to A.Hurwitz (see [Wat, section 15.27]), all roots of the entire function

$$ \Big(\frac{t}{2}\Big)^{-\nu} J_\nu(t) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!\,\Gamma(\nu+l+1)}\Big(\frac{t^2}{4}\Big)^l $$

are real for every $\nu \geq -1$. ($J_\nu(t)$ is the Bessel function of index ν.) For $\nu > -\frac{1}{2}$, the function $(\frac{t}{2})^{-\nu} J_\nu(t)$ admits the integral representation

$$\frac{1}{2}\Big(\frac{t}{2}\Big)^{-\nu} J_\nu(t) = \frac{1}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int\limits_0^1 (1-\xi^2)^{\nu-\frac{1}{2}} \cos t\xi \, d\xi \,. \tag{13.32}$$

Thus, for $\nu > -\frac{1}{2}$ all roots of the entire function $\int\limits_0^1 (1-\xi^2)^{\nu-\frac{1}{2}} \cos t\xi \, d\xi$ are real. If all roots of a real entire function of exponential type are real, then all roots of its derivative are real as well. Thus, for $\nu > -\frac{1}{2}$ all roots of the function $\int\limits_0^1 (1-\xi^2)^{\nu-\frac{1}{2}} \xi \sin t\xi \, d\xi$ are real. However, for $\nu = \frac{p-1}{2}$, the latter function coincides with the function $\frac{1}{p}\ {}^{\mathcal{J}}\Omega(t)$.

For $p = 3$, we do not know whether the function $\mathcal{W}^p_{\partial(B^\infty \times 0)}(t)$ belongs to the Laguerre-Pólya class or not. Our conjecture is that YES. Let us formulate our conjectures in terms of the Fox-Wright functions.

Conjecture 1. *For $0 \leq \lambda \leq 2$, all roots of the Fox-Wright function*

$${}_1\Psi_1 \left\{ \begin{smallmatrix} \frac{1}{2} \\ 1 \end{smallmatrix} ; \begin{smallmatrix} \frac{1}{2} \\ 1+\lambda \end{smallmatrix} ; t \right\} = \sum_{0 \leq k < \infty} \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k}{2}+1+\lambda)} \frac{t^k}{k!} \tag{13.33}$$

lie in the open left half plane.

We proved that Conjecture 1 holds for $0 \leq \lambda \leq 1$ and for $\lambda = 2$.

Conjecture 2. *For $0 \leq \lambda \leq 2$, all roots of the Fox-Wright function*

$${}_1\Psi_2 \left\{ \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} ; \begin{smallmatrix} 1 \\ \frac{1}{2} \end{smallmatrix} \ \begin{smallmatrix} 1 \\ 1+\lambda \end{smallmatrix} ; t \right\} = \sum_{0 \leq l < \infty} \frac{\Gamma(l+1)}{\Gamma(l+\frac{1}{2})\Gamma(l+1+\lambda)} \frac{t^l}{l!} \tag{13.34}$$

are negative and simple.

We proved that Conjecture 2 holds for $0 \leq \lambda \leq 1$, and for $\lambda = 2$.

From Hermite-Biehler theorem it follows that if Conjecture 1 holds for some λ, then also Conjecture 2 holds for this λ.

The conjectures 1 and 2 are related to deep questions related to 'meromorphic multiplier sequences'. (See [CrCs4, Problem 1.1].)

14. Concluding remarks.

1. In the present paper, little use was made of geometric methods. The only general geometric tools used were the Alexandrov-Fenchel inequalities. We did not use the

monotonicity properties of the coefficients belonging to the Steiner polynomials. If V_0, V_1, V_2 are convex sets, such that

$$V_1 \subseteq V_0 \subseteq V_2$$

and $S_{V_0}(t)$, $S_{V_1}(t)$, $S_{V_2}(t)$ are their Steiner polynomials,

$$S_{V_j}(t) = \sum_{0 \leq k \leq n} s_k(V_j)t^k, \quad j = 0,\, 1,\, 2,$$

then for the coefficients of these polynomials the inequalities

$$s_k(V_1) \leq s_k(V_0) \leq s_k(V_2)\,, \quad 0 \leq k \leq n.$$

hold. In this context, the use of the *Kharitonov Criterion for Stability* may be helpful. (Concerning the Kharitonov Criterion see Chapters 5 and 7 of the book [BCK] and the literature quoted there.) The Kharitonov Criterion deals with the 'interval stability' of polynomials. In its simplest form, this criterion allows one to determine whether the polynomial

$$f(t) = \sum_{0 \leq k \leq n} a_k t^k \tag{14.1}$$

with real coefficients a_k is stable, given that these coefficients belong to some intervals:

$$a_k^- \leq a_k \leq a_k^+\,, \quad 0 \leq k \leq n\,. \tag{14.2}$$

Applying this criterion, one needs to construct certain polynomials from the given numbers a_k^-, a_k^+, $0 \leq k \leq n$. (There are finitely many such polynomials.) If all these polynomials are stable, then the arbitrary polynomial $f(t)$, (14.1), whose coefficients satisfy the inequalities (14.2), is stable.

2. In the example of a convex set V, whose Steiner polynomial is not dissipative, the set V is very 'flat' in some direction. (See Theorem 2.12.)

Definition 14.1. The solid convex set V, $V \subset \mathbb{R}^n$, is said to be isotropic (with respect to the point 0), if the integral

$$\int\limits_V |\langle x, e\rangle|^2 dv_n(x)$$

is constant with respect to e, for every vector $e \in \mathbb{R}^n$ such that $\langle e, e\rangle = 1$. Here $\langle\,.\,,\,.\,\rangle$ is the standard scalar product in $\mathbb{R}^n$, and $dv_n(x)$ is the standard n-dimensional differential volume element.

Question. *What can one say about Steiner polynomials associated with a convex set V which is isotropic?*

References

[AdTa] Adler, R.J and J.E. Taylor. *Random Fields and Geometry.* Springer, 2007.

[Al] Alexandrov, A.D. *Selected Works. Vol.1.* Gordon and Breach Publishers, Amsterdam 1996. x+322 pp.

[Al1] Александров, А.Д. *К теории смешанных объёмов выпуклых тел.* II. *Новые неравенства между смешанными объёмами и их применения.* Матем. сборник. **2** (44) (1937), 1205 - 1238.
English transl.:
Alexandrov, A.D. *To the theory of mixed volumes of convex bodies. Part* II. *New inequalities for mixed volumes and their applications.* In: [Al, Pp. 61 - 97].

[Al2] Александров, А.Д. *К теории смешанных объёмов выпуклых тел.* III. *Распространение двух теорем Минковского о выпуклых многогранниках на произвольные выпуклые тела.* Матем. сборник. **3** (45) (1938), 27 - 46.
English transl.:
Alexandrov, A.D. *To the theory of mixed volumes of convex bodies. Part* III. *Extension of two Steiner Theorems on convex polyhedra to all convex bodies.* In: [Al, Pp. 99 - 117].

[Al3] Александров, А.Д. *К теории смешанных объёмов выпуклых тел.* IV. *Смешанные дискриминанты и смешанные объёмы.* Матем. сборник. **3** (45) (1938), 227 - 251. English transl.:
Alexandrov, A.D. *To the theory of mixed volumes of convex bodies. Part* IV. *Mixed discriminants and mixed volumes.* In: [Al, Pp. 119 - 144].

[Arn] Арнольд, В.И. *Комментарии к русскому переводу статьи* [Wey1] (In Russian). [Arnold, V. *Comments to the Russian translation of the paper* [Wey1].] In: [WeyR, pp. 479 - 482].

[Bar] Barnes, E.W. *The asymptotic expansions of integral functions defined by Taylor's series.* Philosoph. Trans. Royal Soc. London. **A 206** (1906), 249 - 297.

[BLD] Beck, M., De Loera, J.A., Develin, M.Pfeifle, J., Stanley, R.P. *Coefficients and roots of Ehrhart polynomals.* In: [IP, pp. 15 - 36]

[BeRo] Beck, M., Robins, S. *Computing the continuous discretely.* Springer, New York. 2007. xviii+226 pp.

[BeGo] Berger, M. Gostiaux, B. *Differential Geometry; Manifolds, Curves, Surfaces.* Springer-Verlag, Berlin·Heidelberg·New York 1988. xii+474 pp.

[BHW] Bey, H., Henk, M., Wills, J. *Notes on the roots of Ehrhart polynomials.* Discrete Comput. Geom., **38**:1 (2007), 81 - 98.

[BCK] Bhattacharyya, S.P., Chapellat, H., Keel, L.H. *Robust Control. The parametric Approach.* Prentice Hall PTR, Upper Saddle River, NJ 1995. xiv+648 p.

[BoFe] Bonneson, T., Fenchel, W. *Theorie der konvexen Körper.* Springer, Berlin, 1934 (In German). English transl.:
Bonneson, T., Fenchel, W. *Theory of Convex Bodies.* BCS Associates, Moscow, Idaho, USA. 1987, x+172 pp. Russian transl.:
Боннезон, Т., Фенхель, В. *Теория выпуклых тел.* Фазис, Москва 2002, x+210.

[Bus] Busemann, H. *Convex Surfaces.* Interscience, New York 1958.

[BuZa] Бураго, Ю.Д, Залгаллер, В.А. *Геометрические неравенства.* Наука, Ленингад 1980. (In Russian.) English translation:
Burago, Yu.D., Zalgaller, V.A. *Geometric inequalities.* Springer-Verlag, Berlin•Heidelberg 1988, xiv+331.

[Cheb] Чеботарев, Н.Г. *О целых функциях с вещественными перемежающимися корнями.* Доклады АН СССР, **35** (1942) 219 - 222. [Chebotarev, N.G. *On entire functions with real interlacing roots.* Doklady AN SSSR, **35** (1942) 219 - 222.] (In Russian.)

[CrCs1] Craven, Th.; Csordas, G. *Location of zeros. Part I. Real polynomials ans entire functions.* Illinois Journ. of Math., **27**:2 (1983), 244 - 278.

[CrCs2] Craven, Th.; Csordas, G., *Jensen polynomials and the Turán and Laguerre inequalities.* Pacif. Journ. of Math., **136**:2 (1989), 241 - 260.

[CrCs3] Craven, Th.; Csordas, G., *Differential operators of infinite order and the distribution of zeros of entire functions.* Jour. Math. Anal. Appl., **186** (1994), pp. 799–820.

[CrCs4] Craven, Th.; Csordas, G. *The Fox-Wright functions and Laguerre multiplier sequences.* Journ. of Math. Analysis and Appl., **314** (2006), pp. 109 - 125.

[CsWi] Csordas, G., Williamson, J. *The zeros of Jensen polynomials are simple.* Proc. Amer. Math. Soc., **49**:1 (1975), 263 - 264.

[deBr] de Branges, L. *Hilbert Spaces of Entire Functions.* Prentice Hall, Englewood Cliffs, N.J., 1968. ix+326 pp.

[Djr] Djrbashian, M.M. *Harmonic Analysis and Boundary Value Problems in the Complex Domain.* (Ser.: Operator Theory Advances and Applications, Vol. **65**.) Birkhauser Verlag, Basel·Boston·Berlin 1993. xiii+256 pp.

[ESV] Edrei, A., Saff, E., Varga, R. *Zeros of Sections of Power Series.* (Lect. Notes in Math, **1002**.) Springer-Verlag, Berlin·Heidelberg·New York 1083, viii+115 pp.

[Fen] Fenchel, W. *Inégalités quadratiques entre des volumes mixtes des corps convexes.* C.R. Acad. Sci. Paris **203** (1936), 647 - 650.

[Fox] Fox, C. *The asymptotic expansion of generalized hypergeometric functions.* Proc. Lond. Math. Soc., **27** (1928), 389 - 400.

[Gant] Гантмахер, Ф.Р. *Теория Матриц. 2-е издание.* Наука, 1966, 575 с. (In Russian). English transl:
Gantmacher, F.R. *The Theory of Matrices.* Chelsea, New York, 1959, 1960.

[GaMi] Gasca, M., Micchelli, Ch.A. (eds) *Total Positivity and its Applications.* Kluver Academic Publishers, Dordrecht·Boston·London 1996, x+518 pp.

[Gra] Gray, A. *Tubes.* 1th ed.: Addison-Wesley 1990,
2nd ed: Birkhäuser Verlag, Basel•Boston•Berlin 2005.
Russian transl. of 1th ed.:
Грей, А. *Трубки.* Мир, Москва 1993.

[Gru] Gruber, P. *Convex and Discrete Geometry.* Springer Verlag, Berlin•Heidelberg 2007. xiv+578 pp.

[EMOT] Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F. *Higher Transcendental Functions. Vol.* III. McGraw-Hill, New York-Toronto-London, 1955. xvii+292 pp.
Russian Translation:

Бейтмен, Г.; Эрдейи, А. *Высшие Трансцендентные Функции.* Том III. (Справочная Математическая Библиотека.) Наука, Москва 1967. 300 сс.

[Had] Hadwiger, H. *Vorlesungen über Inhalt, Öberflache und Isoperimetrie.* Springer, Berlin·Göttingen·Heidelberg 1957, xiii+312 pp. (In German). Russian translation:
Хадвигер. Г. *Лекции об объёме, площади поверхности и изопериметрии.* Наука, Москва, 1966, 416 с.

[HeHe] Henk, M., Hernández Cifre, M.A. *Notes on the roots of Steiner polynomials.* arXiv:mathMG/0703373v1.

[IP] *Integral Points in polyhedra-geometry, number theory, algebra, optimization.* Contemp. Math., **274**. Barvinok, A., Beck, M., Haase, Ch., Reznik, B., Welker, V. - ed. AMS, Provedence, RI, 2005. xii+191.

[Jen] Jensen, J.L.W.V. *Recherches sur la théorie des équations.* Acta Math., **36** (1913), 181 - 195.

[Kar] Karlin, S. *Total Positivity.* Stanford Univ. Press, Stanford CA 1968, xiv+576 pp.

[KaSz] Karlin, S.; Szegö, G. *On certain determinants whose elements are orthogonal polynomials.* Journ. d'Analyse Mathem., **8** (1960), 1 - 157.

[KlRo] Klain, D.A., Rota, G.-C. *Introduction to Geometric Probability.* Cambridge Univ. Press, Cambridge, 1997.

[KrNa] Крейн, М.Г., Неймарк, М.А. *Метод Симметрических и Эрмитовых Форм в Теории Отделения Корней Алгебраических Уравнений.* Государственное Научно Техническое Издательство Украины (ДНТВУ) , Харьков, 1936. 43 с.).
English transl.:
Krein, M.G., Naimark, M.A. *The Method of Symmetric and Hermitian Forms in the Theory of the Separation of the Roots of Algebraic Equations.* Linear and Multilinear Algebra, **10** (1981), pp. 265 - 308.

[Kur] Kurtz, D. *A sufficient condition for all the roots of a polynomial to be real.* Amer. Math. Monthly, **99**:3 (1992), 259 - 263.

[Lag] Laguerre, E. *ŒUVRES. Tome I.* Second edition. Chelsea, New York 1972. xx+468 pp.

[Lag1] Laguerre, E. *Sur quelques points de la théorie des équations numéricques.* Acta Math., **4** (1884), 97 - 120.
Reprinted in [Lag, p. 184 - 206].

[Le] Leichtweiss, K. *Konvexe Mengen.* [*Convex sets* - in German.] Springer, Berlin 1980. 330 s. Russian translation:
Лейхтвейс, К. *Выпуклые множества.* Наука, Москва, 1985. 336 с.

[Lev1] Левин, Б.Я. *Распределение корней целых функций.* ГТТИ, Москва 1956, 632 с.
English transl.: Levin, B.Ja. *Distribution of Zeros of Entire Functions* (Revised Edition). (Transl. of Math. Monogr., Vol. 5.) Amer. Math. Soc., Providence, RI 1980. xii+523 pp.

[Lev2] Levin, B.Ja. *Lectures on Entire Functions.* (Transl. of Mathem. Monogr., **150**.) Amer. Math. Soc., Providence, RI 1996, xv+248 pp.

[Mar1] Marden, M. *Geometry of Polynomials.* 2nd edn. Amer. Math. Soc., Providence, R.I.1966.

[McM] McMullen, P. *Non-linear angle-sum relation for polyhedral cones and polytopes.* Math. Proc. Camb. Phil. Soc., **78** (1975), p. 241 - 251.

[Mil] Milnor, J. *Euler characteristics and finitely additive Steiner measure.* Pp. 213 - 234 in: *John Milnor. Collected papers. Volume 1. Geometry.* Publish or Perish. Huston, Texas, 1994.

[Min] Minkowski, H. *Gesammelte Abhandlungen von Hermann Minkowski, Zweiter Band.* Verlag von B.G. Teubner, Leipzig und Berlin 1911, iv+466 SS.

[Min1] Minkowski, H. *Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs.* In: [Min, S. 131 - 229].

[Min2] Minkowski, H. *Volumen und Oberfläche.* Mathematische Annalen, **57**, S. 447 - 495. Reprinted in [Min, S. 230 - 276].

[Obr] Obreschkoff, N. *Verteilung und Berechnung der Nullstellen Reeller Polynomen.* Deutscher Verlag der Wissenschaften. Berlin 1963, viii+298 ss.

[Ost1] Ostrovskii, I.V. *On zero distribution of sections and tails of power series.* Israel Math. Conference Proc, **15**, 2001, 297 - 310.

[Ost2] Ostrovskii, I. *On a problem of A. Eremenko.* Comput. Methods Func. Theory, **4**:2 (2004), 275 - 282.

[OsPe] Ostrovskii, I.V., Peresyolkova, I.N. *Nonasymptotic results on distribution of zeros of the function* $E_\rho(z,\mu)$. Analysis Mathematica, **23** (1997), 283 - 296.

[Po] Pólya, G. *Collected Papers. Vol. II, Location of Zeros.* (R.P. Boas, ed.) MIT Press, Cambridge, MA, 1974. x+444pp.

[Po1] Pólya, G. *Über die Nullstellen gewisser ganzer Functionen.* [On zeros of certain entire functions.] (In German.) Math. Zeitschrift, **2**, (1918), 352 - 383. Reprinted in [Po, 166 - 197].

[PoSch] Pólya, G., Schur, I. *Über zwei arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen.* Journ. fúr Reine Angew. Mathematik., **144** (1914), 89 - 113. Reprinted in [Po, 100 - 124].

[PoSz] Pólya, G., Szegö, G. *Aufgaben und Lehrsätze aus der Analysis. Zweiter Band* (*Funktionentheorie. Nullstellen. Polynome. Determinanten. Zahlentheorie*). Dritte Auflage: Springer-Verlag,
Berlin·Göttingen·Heidelberg·New York 1964 (In German).
English translation:
Pólya, G., Szegö, G. *Problems and Theorems in Analysis. II.* (*Theory of functions, zeros, polynomials, determinants, number theory, geometry.*) Springer-Verlag, Berlin 1998. xii+392 pp.
Russian translation:
Полиа, Г., Сеге, Г. *Задачи и Теоремы из Анализа. Часть вторая.* (*Теория функций. Распределение нулей. Полиномы. Определители. Теория чисел.*) Третье издание. Наука, Москва 1978. 432 c.

[RaSc] Rahman, Q.L. and G, Schmeisser. *Analytic Theory of Polynomials.* Clarendon Press, Oxford, 2002. xiv+742.

[Schn] Schneider, R. *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge Univ. Press, Cambridge 1993.

[Schu] SCHUR, I.: *Gesammelte Abhandlungen* [*Collected Works*]. *Vol. II.* Springer-Verlag, Berlin·Heidelberg·New York 1973.

[Schu1] SCHUR, I. *Zwei Sätze über algebraische Gleichungen mit lauter reelen Wurzeln* (In German). [*Two theorems on algebraic equations with only real roots.*] Journal für reine und angew. Mathem., **144** (1914), pp. 75 - 88. Reprinted in: [Schu, pp. 56 - 69].

[ShSm] SHEIL-SMALL, T. *Complex Polynomials.* Cambridge Univ. Press, Cambridge 2002.

[St] STEINER, J. *Jacob Steiner's Gesammelte Werke, Zweiter Band.* (K. Weierstrass - Herausgeg.) Verlag von G. Reimer, Berlin 1882.

[Sz] SZEGÖ, G. *Collected Papers. Volume 1: 1915 - 1927.* ASKEY R.-ed. Birkhäuser, Boston·Basel·Stuttgart 1982.

[Sz1] SZEGÖ, G. *Über eine Eigenschaft der Exponentialreihe.* Sitzungsbericht. Berlin. Math. Gesellsch., **23** (1924), 50 - 64. Reprinted in [Sz, pp. 646 - 660].

[St1] STEINER, J. *Über parallele Flächen* Monatsbericht der Akademie der Wissenschaften zu Berlin (1840), 114 - 118. Reprinted in [St], 172 - 176.

[Tei] TEISSIER, B. *Bonneson-type inequalities in algebraic geometry, I: Introduction to the problem.* Seminar on Differential Geometry, pp. 85 - 105. (Ann. of Math. Study, **102**.) Princeton Univ. Press. Princeton, N.J. 1982.

[Tit] TITCHMARSH, E.C. *The Theory of Functions.* Oxford University Press. Oxford 1939. x+454 p.

[Wat] WATSON, G.N. *A Treatise on the Theory of Bessel Functions. Second ed.* Cambridge Univ. Press, Cambridge, 1944. vi+804 pp.

[Wat1] WATSON, G.N. *A class of integral functions defined by Taylor's series.* Trans. Cambr. Philoc. Soc., **22** (1913), 15 - 37.

[Web] WEBSTER, R. *Convexity.* Oxford University Press, Oxford·New York·Tokyo 1994, xviii+444 pp.

[Wey] WEYL, H. *Gesammelte Abhandlungen. Band III.* — Chandrasekharan - ed. Springer-Verlag, Berlin·Heidelberg·New York 1968.

[WeyR] ВЕЙЛЬ, Г., *Избранные Труды.* Арнольд, В.И.; Паршин, А.Н. — изд., Наука, Москва 1984. 512 сс.(Russian).
[WEYL, H., *Selected Papers*, Arnold, V.I.; Parshin, A.N. — eds., Nauka, Moskow 1984, 512 pp.]

[Wey1] WEYL, H. *On the volume of tubes.* Amer. Journ of Math., **61** (1939), 461 - 472. Reprinted in [Wey, pp. 658 - 670].
Russian transl.:
ВЕЙЛЬ, Г. *Об объёме труб*, [WeyR, сс. 246 - 255].

[WhWa] WHITTAKER, E.T., WATSON, G.N. *A Course of Modern Analysis.* Cambridge Univ. Press, Cambridge 1927. vi+608 pp.
Russian transl.:
УИТТЕКЕР, Э.Т., ВАТСОН, Дж.Н. *Курс Современного Анализа. Том 2.* Физматгиз, Москва 1963. 515 с.

[Wr1] WRIGHT, E.M. *The asymptotic expansion of the generalized hypergeometric functions.* Journ. London Math. **10** (1935), 286 - 293.

[Wr2] Wright, E.M. *The asymptotic expansion of integral functions defined by Taylor series.* Philosoph. Trans. Royal Soc. London. **A 238**, Issue 795 (1940), 423 - 451.

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